Complete Reference for Odd Graceful Labeling of Cyclic Snakes

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Abstract:

A graph is called odd graceful if it has an odd graceful labeling. The definition of odd graceful graphs was introduced by Gnanajothi [1]. Here we will define the graph (m, k) C_4-snake and prove that the graphs kC_4- snake is odd graceful, we prove the graph $(2, k)$ C_4-snake is odd graceful, we will introduce the odd graceful labeling of the graph $(3, k)$ C_4-snake, we will introduce the odd graceful labeling of the graph (m, k) C_4-snake**.** We prove that the graphs kC_6- snake is odd graceful, we prove the graph $(2, k)$ C_6-snake is odd graceful, we will introduce the odd graceful labeling of the graph $(3, k)$ C_6-snake, we will introduce the odd graceful labeling of the graph (m, k) C_6-snake**.** we will prove that the graph $(2, k)$ C_8-snake is odd graceful, we will introduce the odd graceful labeling of the graph $(3, k)$ C_8-snake, we will introduce the odd graceful labeling of the graph (m, k) C_8-snake, and after that we will prove that the graph kC_{12}-snake is odd graceful.

Introduction:

A graph is called odd graceful if it has an odd graceful labeling. Gnanajothi [1] introduced the definition of odd graceful graphs. A graph G with q edges is said to be an odd graceful if there is an injection ϕ from the vertices of G "$V(G)$" to the set $\{0, 1, 2\ldots2q - 1\}$ such that, when each edge $\{u, v\}$ is assigned the label $k = |\phi(u) - \phi(v)|$, $k \in \{1, 3, 5,\ldots 2q - 1\}$.

The graphs considered here will be finite, undirected and simple. We denote the vertex sets and the edges sets of a graph G by $V(G)$ and $E(G)$ respectively.

A cycle in a graph G is a closed walk of the form $w_1w_2w_3\ldots w_k$ where $k \geq 3$, and if for some $i \neq j$ we have $w_i = w_j$ then $\{i, j\} = \{1, 2\}$. A kC_n-snake is a connected graph with k blocks; each of the blocks is isomorphic to the cycle C_n, such that the block-cut-vertex graph is a path. We also call a kC_n-snake as a cyclic snake. The graph kC_n-snake was introduced by Barrientos as generalization of the concept of triangular snake introduced by Rosa [2]. Rosa [2] The cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4), E. M. Badr, M. I. Moussa and K. Kathiresan [3] proved that The cycle C_n is odd graceful if n is even, $n \geq 4$. Now we define The graphs mC_4 as the family of graphs consisting of m copies of C_4 with two non adjacent vertices in common

Example 1

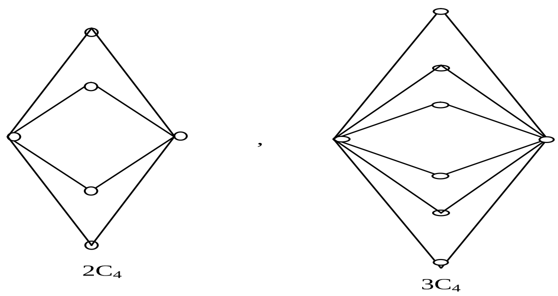

Figure1: The graphs 2C_4 & 3C_4.

and The graphs (m, k) C_4 as the family of graphs kC_4-snake where every block has m copies of C_4 with two non adjacent vertices in common. such that the number of blocks is denoted by k.

Example 2

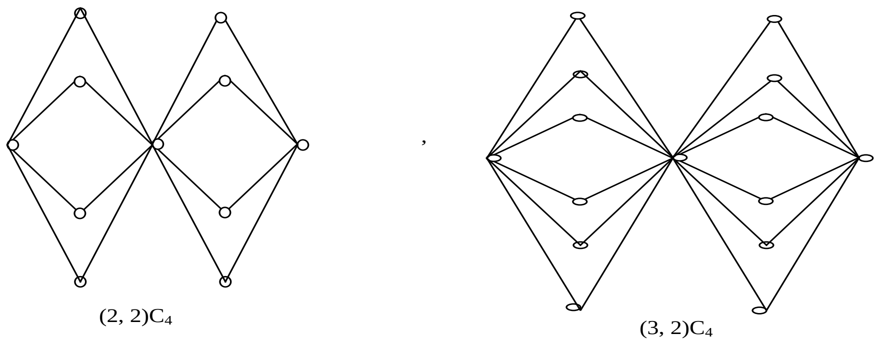

Figure 2: The graphs (2, 2) C_4 & (3,2) C_4.

Main Results:

Theorem 1: The graph kC_4-snake " (1, k) C_4-snake" is odd graceful.

Proof

Let kC_4-snake be a graph and has q edges. Let $u_1u_2u_3\ldots u_{k+1}$, $w_1w_2w_3\ldots w_k$ and $v_1v_2v_{3\ldots}v_k$ are the vertices of kC_4-snake, such that v_i & w_i are put between u_i and $u_{i+1,}$ $i = 1, 2, 3\ldots k$, the graph kC_4- snake has number of edges "q" $4k$, as shown in the next figure.

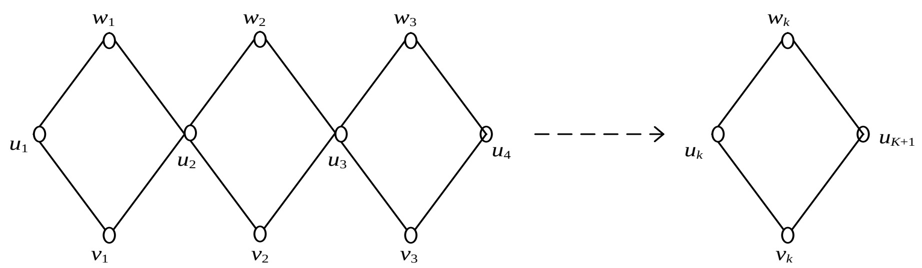

Figure 3: The graph kC_4-snake

The number of edges" q"= $4k$

Define $\phi: V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 4i - 4$, $i = 1, 2, 3\ldots k + 1$

$\phi(w_i) = 2q - 4i+3$, $i = 1, 2, 3\ldots k$

$\phi(v_i) = 2q - 4i + 1$, $i = 1, 2, 3\ldots k$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_4$-snake

$$\exists \max_{v \in V(G)} \phi(v) = \max\{ \max_{1 \le i \le k+1} (4i - 4), \max_{1 \le i \le k}(2q - 4i + 3), \max_{1 \le i \le k}(2q - 4i + 1)\}$$

$$= \max\{\max\{0, 4, 8 \ldots 4k\}, \max\{2q - 1, 2q - 5 \ldots 2q - 4k + 3\}, \max\{2q - 3, 2q - 7 \ldots 2q - 4k + 1\}\}$$

$$= 2q - 1$$

Hence, we find that :−

$$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $|\phi(w_i) - \phi(u_i)| = \{2q - 8i + 7 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 9 \ldots 2q - 8k + 7\} \quad , 2q = 8k$

$= \{2q - 1, 2q - 9 \ldots 7\}$

The range of $|\phi(w_i) - \phi(u_{i+1})| = \{2q - 8i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 13 \ldots 2q - 8k + 3\} \quad , 2q = 8k$

$= \{2q - 5, 2q - 13 \ldots 3\}$

The range of $|\phi(v_i) - \phi(u_i)| = \{2q - 8i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 11 \ldots 2q - 8k + 5\} \quad , 2q = 8k$

$= \{2q - 3, 2q - 11 \ldots 5\}$

The range of $|\phi(v_i) - \phi(u_{i+1})| = \{2q - 8i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q - 15 \ldots 2q - 8k + 1\} \quad , 2q = 8k$

$= \{2q - 7, 2q - 15 \ldots 1\}$

Hence, $\{\, |\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph kC_4-snake is odd graceful.

Example 3

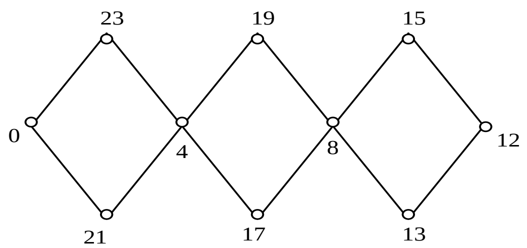

Figure 4: The odd graceful labeling of the graph 3C_4-snake

Theorem 2: The graph (2, k) C_4-snake is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, $w_1^1w_1^2w_2^1w_2^2\ldots.w_k^1w_k^2$ and $v_1^1v_1^2v_2^1v_2^2\ldots.v_k^1v_k^2$ are the vertices of (2, k) C_4-snake, such that v_i^j & w_i^j are put between u_i and u_{i+1}, i = 1, 2, 3… k, j = 1, 2. v_i^j is beneath v_i^{j+1} and w_i^j is beneath w_i^{j+1} where j = 1, 2, the graph (2, k) C_4-snake has number of edges "q" 8k, as shown in the next figure. .

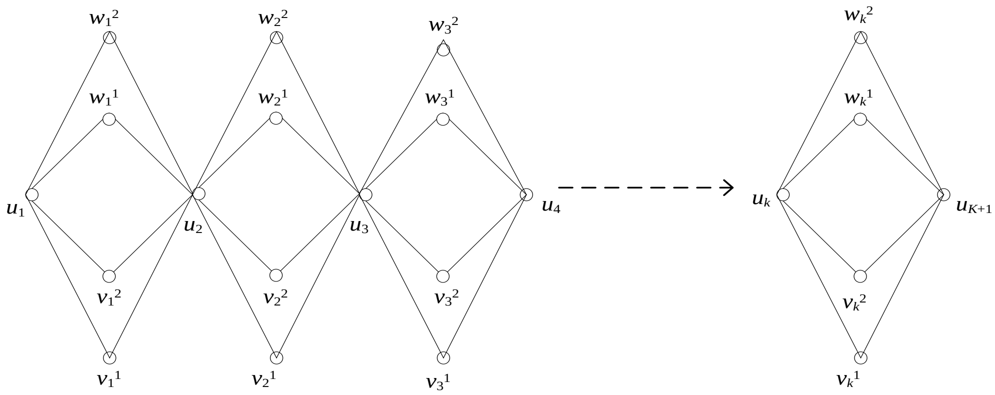

Figure 5: The graph (2, *k*) C_4-snake

The number of edges" q"= 8k

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 8i - 8$, i = 1, 2, 3… k + 1

$\phi(w_i^j) = 2q - 8i + 2j + 3$, i = 1, 2, 3… k , j = 1, 2

$\phi(v_i^j) = 2q - 8i + 2j - 1$, i = 1, 2, 3… k , j = 1, 2

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (2, k) C_4-snake

$\exists \max_{v \in V(G)} \phi(v) =$

$$\max\{\max_{1\le i\le k+1}(8i-8), \max_{\substack{1\le i\le k,\\ j=1,2}}(2q-8i+2j+3), \max_{\substack{1\le i\le k,\\ j=1,2}}(2q-8i+2j-1)\}$$

$$= \max\{\max\{0, 8, 16 \dots 8k\}, \max_{j=1,2}\{2q+2j-5, 2q+2j-13 \dots 2q+2j-8k+3\}, \max_{j=1,2}\{2q+2j-9, 2q+2j-17 \dots 2q+2j-8k-1\}\}$$

$$= 2q - 1$$

Hence, we find that :-

$$\phi(v) \in \{0, 1, 2 \dots 2q-1\}$$

• It's obvious that $\forall\ u, v \in V(G) \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \dots 2q-1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\dots 2q-1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \dots 2q-1\}$ and that's as following:

The range of $|\phi(w_i^1) - \phi(u_i)| = \{2q-16i+13 \quad , i = 1, 2 \dots k\}$

$= \{2q-3, 2q-19 \dots 2q-16k+13\} \quad , 2q = 16k$

$= \{2q-3, 2q-19 \dots 13\}$

The range of $|\phi(w_i^1) - \phi(u_{i+1})| = \{2q-16i+5 \quad , i = 1, 2 \dots k\}$

$= \{2q-11, 2q-27 \dots 2q-16k+5\} \quad , 2q = 16k$

$= \{2q-11, 2q-27 \dots 5\}$

The range of $|\phi(w_i^2) - \phi(u_i)| = \{2q-16i+15 \quad , i = 1, 2 \dots k\}$

$= \{2q-1, 2q-17 \dots 2q-16k+15\} \quad , 2q = 16k$

$= \{2q-1, 2q-17 \dots 15\}$

The range of $| \phi(w_i^2) - \phi(u_{i+1}) | = \{2q - 16i + 7 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 9, 2q - 25 \ldots 2q - 16k + 7 \} \quad , 2q = 16k$

$= \{2q - 9, 2q - 25 \ldots 7\}$

The range of $| \phi(v_i^1) - \phi(u_i) | = \{2q - 16i + 9 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 7, 2q - 23 \ldots 2q - 16k + 9 \} \quad , 2q = 16k$

$= \{2q - 7, 2q - 23 \ldots 9\}$

The range of $| \phi(v_i^1) - \phi(u_{i+1}) | = \{2q - 16i + 1 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 15, 2q - 31 \ldots 2q - 16k + 1 \} \quad , 2q = 16k$

$= \{2q - 15, 2q - 31 \ldots 1\}$

The range of $| \phi(v_i^2) - \phi(u_i) | = \{2q - 16i + 11 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 5, 2q - 21 \ldots 2q - 16k + 11 \} \quad , 2q = 16k$

$= \{2q - 5, 2q - 21 \ldots 11\}$

The range of $| \phi(v_i^2) - \phi(u_{i+1}) | = \{2q - 16i + 3 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 13, 2q - 29 \ldots 2q - 16k + 3 \} \quad , 2q = 16k$

$= \{2q - 13, 2q - 29 \ldots 3\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\,v \in E(G)\} = \{ 1, 3, 5 \ldots 2q - 1\}$.

So the graph $(2, k)$ C_4-snake is odd graceful.

Example 4

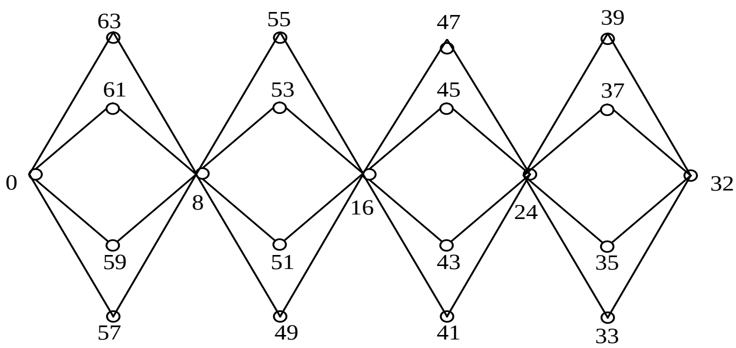

Figure 6: The odd graceful labeling of the graph (2, 4) C_4-snake

Theorem 3: The graph (3, k) C_4-snake is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, $w_1^1w_1^2w_1^3w_2^1w_2^2w_2^3\ldots.w_k^1w_k^2w_k^3$ and $v_1^1v_1^2v_1^3v_2^1v_2^2v_2^3\ldots.v_k^1v_k^2v_k^3$ are the vertices of (3, k) C_4-snake, such that v_i^j & w_i^j are put between u_i and $u_{i+1,}$ i = 1, 2, 3… k, j = 1, 2, 3. v_i^j is beneath v_i^{j+1} and w_i^j is beneath w_i^{j+1} where j = 1, 2, 3, the graph (3, k) C_4-snake has number of edges "q" $12k$, as shown in the next figure.

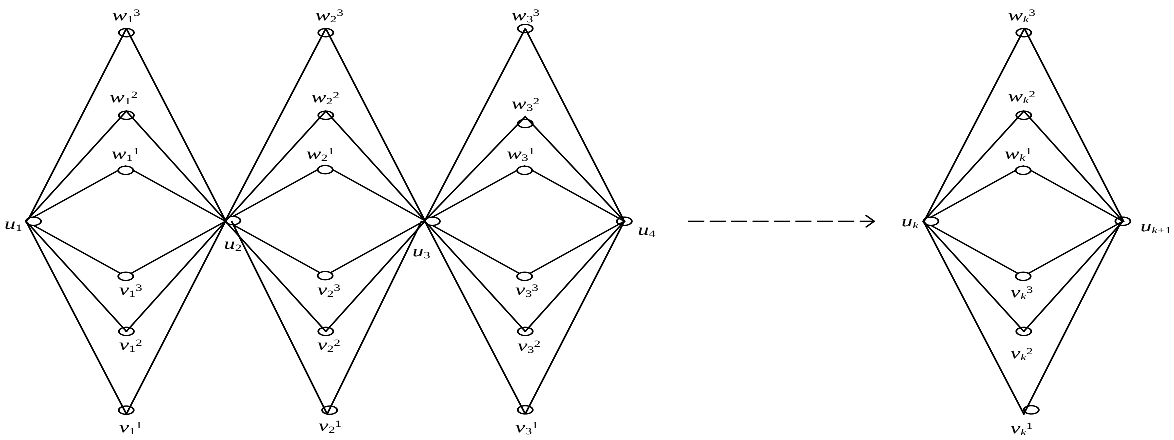

Figure 7: The graph (3, k) C_4-snake

The number of edges" q"= $12k$

Define $\phi: V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 12i - 12$, i = 1, 2, 3… k + 1

$\phi(w_i^j) = 2q - 12i + 2j + 5$, i = 1, 2, 3… k , j= 1, 2, 3

$\phi(v_i^j) = 2q - 12i + 2j - 1$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (3, k)$ C_4-snake

$\exists$ $\max_{v \in V(G)} \phi(v)$

$$=\max\{ \max_{1\le i\le k+1}(12i-12), \max_{\substack{1\le i\le k,\\ 1\le j\le 3}}(2q-12i+2j+5), \max_{\substack{1\le i\le k,\\ 1\le j\le 3}}(2q-12i+2j-1)\}$$

$$= \max\{\max\{0, 12, 24 \ldots 12k\}, \max_{j=1,2,3}\{2q+2j-7, 2q+2j-19 \ldots 2q+2j-12k+5\}, \max_{j=1,2,3}\{2q+2j-13, 2q+2j-25 \ldots 2q+2j-12k-1\}\}$$

$= 2q - 1$

Hence, we find that :-

$\phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set { 0,1,2…$2q$ - 1}

• Now, we want to show that the labels of the edges of G belong to the set {1, 3, 5 … $2q$ - 1 } and that's as following:

The range of $|\phi(w_i^1) - \phi(u_i)| = \{2q - 24i + 19$, $i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 29 \ldots 2q - 24k + 19\}$, $2q = 24k$

$= \{2q - 5, 2q - 29 \ldots 19\}$

The range of $|\phi(w_i^1) - \phi(u_{i+1})| = \{2q - 24i + 7$, $i = 1, 2 \ldots k\}$

$= \{2q - 17, 2q - 41 \ldots 2q - 24k + 7\}$, $2q = 24k$

$= \{2q - 17, 2q - 41 \ldots 7\}$

The range of $|\phi(w_i^2) - \phi(u_i)| = \{2q - 24i + 21 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 27 \ldots 2q - 24k + 21\} \quad , 2q = 24k$

$= \{2q - 3, 2q - 27 \ldots 21\}$

The range of $|\phi(w_i^2) - \phi(u_{i+1})| = \{2q - 24i + 9 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 15, 2q - 39 \ldots 2q - 24k + 9\} \quad , 2q = 24k$

$= \{2q - 15, 2q - 39 \ldots 9\}$

The range of $|\phi(w_i^3) - \phi(u_i)| = \{2q - 24i + 23 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 25 \ldots 2q - 24k + 23\} \quad , 2q = 24k$

$= \{2q - 1, 2q - 25 \ldots 23\}$

The range of $|\phi(w_i^3) - \phi(u_{i+1})| = \{2q - 24i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 13, 2q - 37 \ldots 2q - 24k + 11\} \quad , 2q = 24k$

$= \{2q - 13, 2q - 37 \ldots 11\}$

The range of $|\phi(v_i^1) - \phi(u_i)| = \{2q - 24i + 13 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 11, 2q - 35 \ldots 2q - 24k + 13\} \quad , 2q = 24k$

$= \{2q - 11, 2q - 35 \ldots 13\}$

The range of $|\phi(v_i^1) - \phi(u_{i+1})| = \{2q - 24i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 23, 2q - 47 \ldots 2q - 24k + 1\} \quad , 2q = 24k$

$= \{2q - 23, 2q - 47 \ldots 1\}$

The range of $|\phi(v_i^2) - \phi(u_i)| = \{2q - 24i + 15 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 9, 2q - 33 \ldots 2q - 24k + 15\} \quad , 2q = 24k$

$= \{2q - 9, 2q - 33 \ldots 15\}$

The range of $| \phi(v_i^2) - \phi(u_{i+1}) | = \{2q - 24i + 3 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 21, 2q - 35 \ldots 2q - 24k + 3 \} \qquad , 2q = 24k$

$= \{2q - 21, 2q - 35 \ldots 3\}$

The range of $| \phi(v_i^3) - \phi(u_i) | = \{2q - 24i + 17 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 7, 2q - 31 \ldots 2q - 24k + 17 \} \qquad , 2q = 24k$

$= \{2q - 7, 2q - 31 \ldots 17\}$

The range of $| \phi(v_i^3) - \phi(u_{i+1}) | = \{2q - 24i + 5 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 19, 2q - 43 \ldots 2q - 24k + 5 \} \qquad , 2q = 24k$

$= \{2q - 9, 2q - 43 \ldots 5\}$

Hence, $\{ | \phi(u) - \phi(v) | : uv \in E(G)\} = \{ 1, 3, 5 \ldots 2q - 1\}$.

So the graph $(3, k)$ C_4-snake is odd graceful.

Example 5

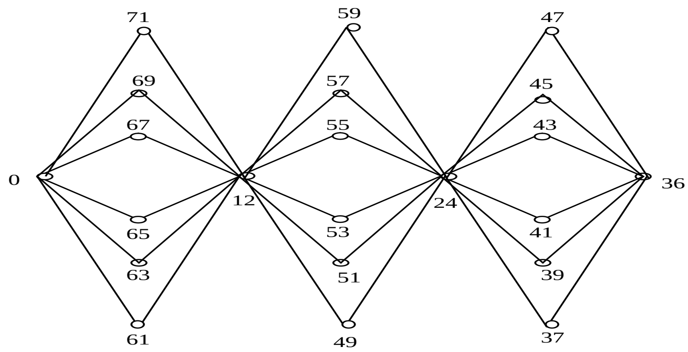

Figure 8: The odd graceful labeling of the graph (3, 3) C_4-snake

Theorem 4: The graph (m, k) C_4-snake is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, w_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2, 3,\ldots m$ and v_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2, 3\ldots m$ are the vertices of (m, k) C_4-snake, such that v_i^j & w_i^j are put between u_i and u_{i+1}, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3\ldots m$. v_i^j is beneath v_i^{j+1} and w_i^j is beneath w_i^{j+1} where

j = 1, 2, 3…m, the graph (m, k) C_4-snake has number of edges "q" $4m\ k$, as shown in the next figure.

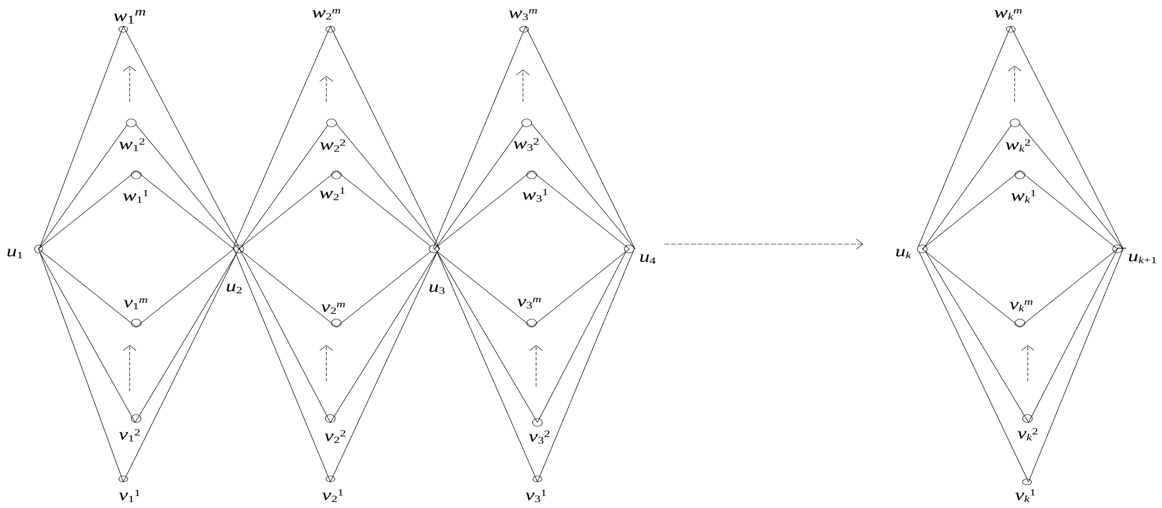

Figure 9: The graph (m, k) C_4-snake

The number of edges" q" = $4m\ k$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots\ 2q\text{-}1\}$ as following:

$\phi\,(u_i) = 4m\,(i - 1)$, $i = 1, 2, 3\ldots\ k + 1$

$\phi\,(w_i^{\,j}) = 2q - 4mi + 2j + 2m - 1$, $i = 1, 2, 3\ \ldots\ k$, $j = 1, 2, 3\ldots\ m$

$\phi\,(v_i^{\,j}) = 2q - 4mi + 2j - 1$, $i = 1, 2, 3\ \ldots\ k$, $j = 1, 2, 3\ldots m$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (m, k) C_4-snake

$\exists$ $$\max_{v \in V(G)} \phi\,(v) =$$

$$\max\{\max_{1 \le i \le k+1}(4mi - 4m),\ \max_{\substack{1 \le i \le k \\ 1 \le j \le m}}(2q - 4mi + 2j + 2m - 1)$$

$$,\ \max_{\substack{1 \le i \le k, \\ 1 \le j \le m}}(2q - 4mi + 2j - 1)\}$$

$= \max\{\max\{0, 4m, 8m \ldots 4mk\},$

$\max_{1\le j\le m} \{2q + 2j - 2m - 1, 2q + 2j - 6m - 1 \ldots 2q + 2j - 4mk - 1\}$

,

$\max_{j=1,2,3} \{2q + 2j - 4m - 1, 2q + 2j - 8m - 1 \ldots 2q + 2j - 4mk - 1\}$

$\}$

$= 2q - 1$

Hence, we find that :–

$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

• It's obvious that $\forall\, u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $|\phi(w_i^j) - \phi(u_i)| = \{2q - 8mi + 2j + 6m - 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

$= \{2q + 2j - 2m - 1, 2q + 2j - 10m - 1 \ldots 2q + 2j - 8mk + 6m - 1, j = 1, 2, 3 \ldots m\}$, $2q = 8mk$

The range of $|\phi(w_i^j) - \phi(u_{i+1})| = \{2q - 8mi + 2j + 2m - 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

$= \{2q + 2j - 6m - 1, 2q + 2j - 14m - 1 \ldots 2q + 2j - 8mk + 2m - 1, j = 1, 2, 3 \ldots m\}$, $2q = 8mk$

The range of $|\phi(v_i^j) - \phi(u_i)| = \{2q - 8mi + 2j + 4m - 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

$= \{2q + 2j - 4m - 1, 2q + 2j - 12m - 1 \ldots 2q + 2j - 8mk + 4m - 1, j = 1, 2, 3 \ldots m\}$, $2q = 8mk$

The range of $|\phi(v_i^j) - \phi(u_{i+1})| = \{2q - 8mi + 2j - 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

$= \{2q + 2j - 8m - 1, 2q + 2j - 16m - 1 \ldots 2q + 2j - 8mk$
$- 1 , \ j = 1, 2, 3 \ldots m \}$, $2q = 8mk$

Hence, $\{ \mid \phi(u) - \phi(v) \mid : u\,v \in E(G)\} = \{ 1, 3, 5 \ldots 2q - 1\}$.

So the graph (m, k) C_4-snake is odd graceful.

Example 6

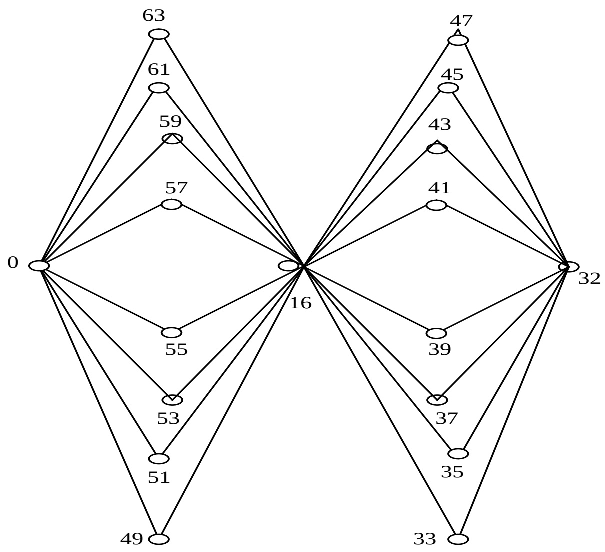

Figure 10: The odd graceful labeling of the graph (4, 2) C_4-snake

Theorem 1: The graph kC_6 " $(1, k)$ C_6" is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, $w_1w_2w_3\ldots w_k$, $v_1v_2v_3\ldots v_k$ and $w_{11}w_{12}w_{21}$ $w_{22}w_{31}w_{32}$ $\ldots w_{k1}w_{k2}$ are the vertices of kC_6, such that v_i & w_i are put between u_i and u_{i+1}, w_{il} is put between w_p and u_i such that $i = 1, 2, 3\ldots k$, $p = 1,2,3\ldots k$, $l = 1, 2$, the graph kC_6 has number of edges "q" $6k$, as shown in the next figure.

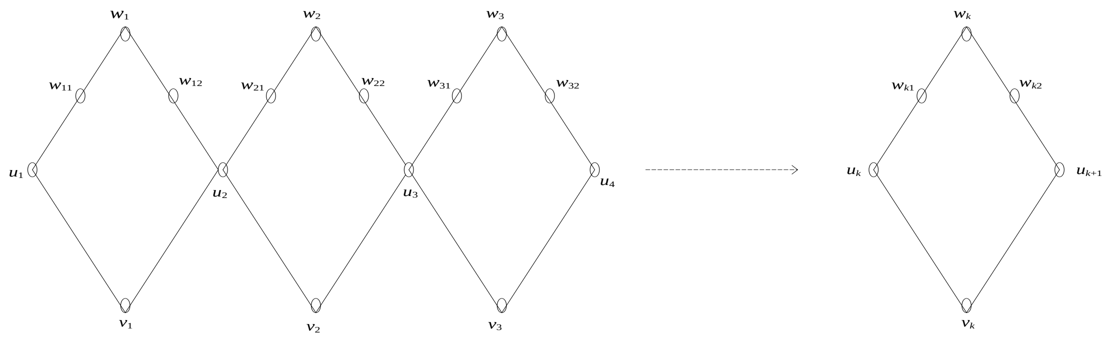

Figure 3: The graph kC_6

The number of edges" q"= $6k$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 4i - 4 \qquad , i = 1, 2, 3\ldots k + 1$

$\phi(w_i) = 4i - 2 \qquad , i = 1, 2, 3\ldots k$

$\phi(w_{il}) = 2q - 4i - 2l + 5 \qquad , i = 1, 2, 3\ldots k , l = 1, 2$

$\phi(v_i) = 4k - 4i + 1 \qquad , i = 1, 2, 3\ldots k$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_6$

$$\exists \quad \max_{v \in V(G)} \phi(v) = \max\{ \max_{1 \le i \le k+1} (4i - 4), \max_{1 \le i \le k} (4i - 2), \max_{\substack{1 \le i \le k \\ l = 1,2}} (2q - 4i - 2l + 5), \max_{1 \le i \le k} (4k - 4i + 1)\}$$

$$= \max\{\max\{0, 4, 8 \ldots 4k\}, \max\{2, 6 \ldots 4k - 2\}, \max_{l=1,2}\{2q - 2l + 1, 2q - 2l - 3 \ldots 2q - 2l - 4k + 5\}, \max\{4k - 3, 4k - 7 \ldots 1\}\}$$

$$= 2q - 1$$

Hence, we find that :−

$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2 \ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $|\phi(w_{i1}) - \phi(u_i)| = \{2q - 8i + 7 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 9 \ldots 2q - 8k + 7\} \quad , 2q = 12k$

The range of $|\phi(w_{i1}) - \phi(w_i)| = \{2q - 8i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 11 \ldots 2q - 8k + 5\} \quad , 2q = 12k$

The range of $|\phi(w_{i2}) - \phi(w_i)| = \{2q - 8i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 13 \ldots 2q - 8k + 3\} \quad , 2q = 12k$

The range of $|\phi(w_{i2}) - \phi(u_{i+1})| = \{2q - 8i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q - 9 \ldots 2q - 8k + 1\} \quad , 2q = 12k$

The range of $|\phi(v_i) - \phi(u_i)| = \{4k - 8i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{4k - 3, 4k - 11 \ldots 5 - 4k\} \quad , 2q = 12k$

The range of $|\phi(v_i) - \phi(u_{i+1})| = \{4k - 8i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{4k - 7, 4k - 15 \ldots 1 - 4k\} \quad , 2q = 12k$

Hence, $\{\ |\phi(u) - \phi(v)| : u\,v \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph kC_6 is odd graceful.

Example 3

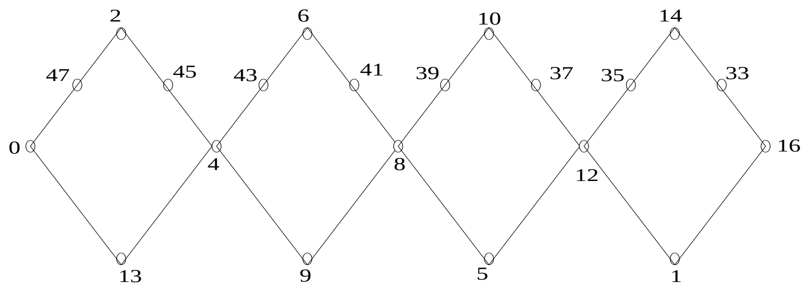

Figure 4: The odd graceful labeling of the graph 4C_6

Theorem 2: The graph (2, k) C_6 is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, $w_1^1w_1^2w_2^1w_2^2\ldots.w_k^1w_k^{\ 2}$, $v_1^1v_1^2v_2^1v_2^2\ldots.v_k^1v_k^{\ 2}$ and w_{il}^j, where i = 1, 2, 3…k , l = 1, 2, j = 1, 2, are the vertices of (2, k) C_6, such that v_i^j & w_i^j are put between u_i and $u_{i+1,}$ w_{i1}^j between u_i and w_p^j, such that i = 1, 2, 3…k+1, p = 1, 2, 3…k , l = 1, 2, j = 1, 2, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1} and w_{i1}^j is beneath w_{i1}^{j+1} where j = 1, 2, the graph (2, k) C_6 has number of edges "q" 12k, as shown in the next figure.

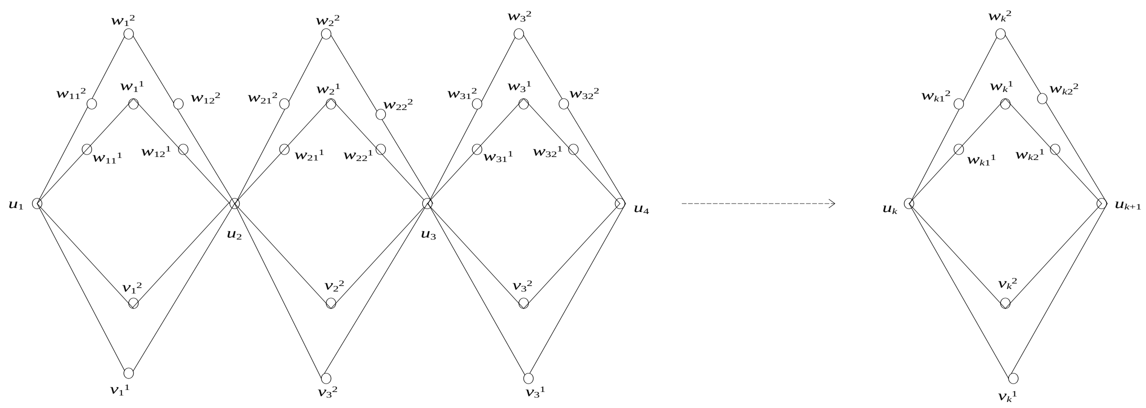

Figure 5: The graph (2, k) C_6

The number of edges" q"= 12k

Define ϕ: $V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 8i - 8$, i = 1, 2, 3… k+1

$\phi(w_i^j) = 8i + 4j - 10$, i = 1, 2, 3… k , j = 1, 2

$\phi(w_{il}^j) = 2q - 8i - 4l + 2j + 7$, i = 1, 2, 3… k , l = 1, 2, j = 1, 2

$\phi(v_i^j) = 8k - 8i + 2j - 1$, $i = 1, 2, 3 \ldots k$, $j = 1, 2$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (2, k)\ C_6$

$$\exists \quad \max_{v \in V(G)} \phi(v) =$$

$$\max\{ \max_{1 \le i \le k+1} (8i - 8),\ \max_{\substack{1 \le i \le k,\\ j=1,2}} (8i + 4j - 10),$$

$$\max_{\substack{1 \le i \le k\\ l=1,2,\\ j=1,2}} (2q - 8i - 4l + 2j + 7),\ \max_{\substack{1 \le i \le k,\\ j=1,2}} (8k - 8i + 2j - 1)\}$$

$$= \max\{\max\{0, 8, 16 \ldots 8k\},$$

$$\max_{j=1,2}\{4j - 2, 4j + 6 \ldots 4j + 8k - 10\},$$

$$\max_{\substack{l=1,2,\\ j=1,2}}\{2q - 4l + 2j - 1, 2q - 4l + 2j - 9 \ldots 2q - 4l + 2j - 8k + 7\}$$

$$, \max_{j=1,2}\{8k + 2j - 9,\ 8k + 2j - 17 \ldots 2j - 1\}\}$$

$$= 2q - 1$$

Hence, we find that :−

$$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set { 0,1,2…2q - 1}

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $|\phi(w_{i1}^1) - \phi(u_i)| = \{2q - 16i + 13$, $i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 19 \ldots 2q - 16k + 13\}$, $2q = 24k$

The range of $|\phi(w_{i1}^1) - \phi(w_i^1)| = \{2q - 16i + 11$, $i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 21 \ldots 2q - 16k + 11\}$, $2q = 24k$

The range of $|\phi(w_{i1}^2) - \phi(u_i)| = \{2q - 16i + 15 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 17 \ldots 2q - 16k + 15\} \quad , 2q = 24k$

The range of $|\phi(w_{i1}^2) - \phi(w_i^2)| = \{2q - 16i + 9 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q - 23 \ldots 2q - 16k + 9\} \quad , 2q = 24k$

The range of $|\phi(w_{i2}^1) - \phi(u_{i+1})| = \{2q - 16i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 15, 2q - 31 \ldots 2q - 16k + 1\} \quad , 2q = 24k$

The range of $|\phi(w_{i2}^1) - \phi(w_i^1)| = \{2q - 16i + 7 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 9, 2q - 25 \ldots 2q - 16k + 7\} \quad , 2q = 24k$

The range of $|\phi(w_{i2}^2) - \phi(u_{i+1})| = \{2q - 16i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 13, 2q - 29 \ldots 2q - 16k + 3\} \quad , 2q = 24k$

The range of $|\phi(w_{i2}^2) - \phi(w_i^2)| = \{2q - 16i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 11, 2q - 27 \ldots 2q - 16k + 5\} \quad , 2q = 24k$

The range of $|\phi(v_i^1) - \phi(u_i)| = \{8k - 16i + 9 \quad , i = 1, 2 \ldots k\}$

$= \{8k - 7, 8k - 23 \ldots 9 - 8k\} \quad , 2q = 24k$

The range of $|\phi(v_i^2) - \phi(u_i)| = \{8k - 16i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{8k - 5, 8k - 21 \ldots 11 - 8k\} \quad , 2q = 24k$

The range of $|\phi(v_i^1) - \phi(u_{i+1})| = \{8k - 16i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{8k - 15, 8k - 31 \ldots 1 - 8k\} \quad , 2q = 24k$

The range of $|\phi(v_i^2) - \phi(u_{i+1})| = \{8k - 16i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{8k - 13, 8k - 29 \ldots 3 - 8k\} \quad , 2q = 24k$

Hence, $\{ |\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph(2, k) C_6 is odd graceful.

Example 4

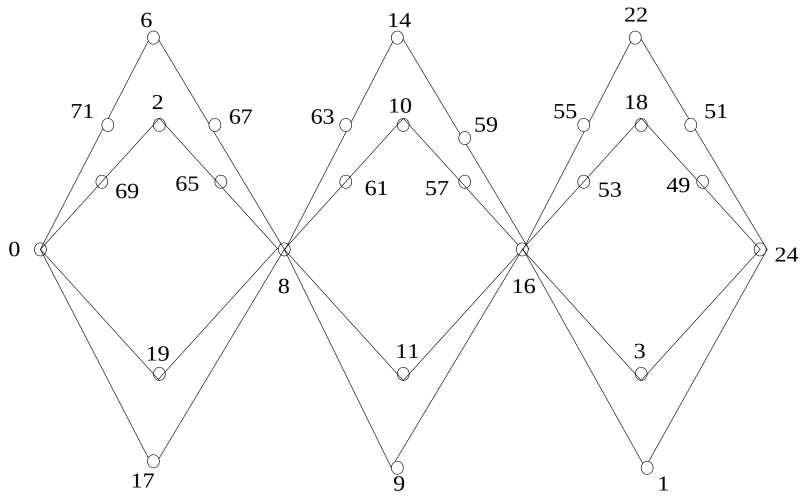

Figure 6: The odd graceful labeling of the graph (2, 3) C_6

Theorem 3: The graph (3, k) C_6 is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, w_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$, v_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$ and w_{il}^j, where $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2, 3$, are the vertices of (3, k) C_6, such that v_i^j & w_i^j are put between u_i and u_{i+1}, w_{i1}^j between u_i and w_p^j, such that $i = 1, 2, 3\ldots k+1$, $p = 1,2,3\ldots k$, $l = 1, 2$, $j = 1, 2, 3$, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1} and w_{i1}^j is beneath w_{i1}^{j+1} where $j = 1, 2, 3$, the graph (3, k) C_6 has number of edges "q" $18k$, as shown in the next figure.

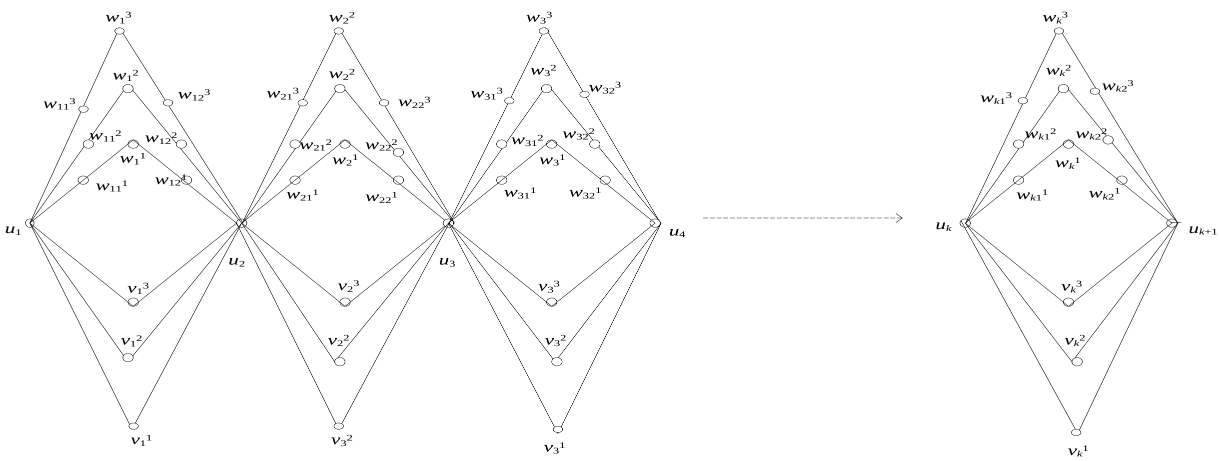

Figure 7: The graph (3, k) C_6

The number of edges" q"= 18k

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 12i - 12$, $i = 1, 2, 3\ldots k+1$

$\phi(w_i^j) = 12i + 4j - 14$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$

$\phi(w_{il}^j) = 2q - 12i - 6l + 2j + 11$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2, 3$

$\phi(v_i^j) = 12k - 12i + 2j - 1$, $i = 1, 2, 3\ldots$ k , $j = 1, 2, 3$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (3, k) C_6

$$\exists \max_{v \in V(G)} \phi(v) = \max\{ \max_{1 \le i \le k+1} (12i - 12), \max_{\substack{1 \le i \le k,\\ j=1,2,3}} (12i + 4j - 14), \max_{\substack{1 \le i \le k\\ l=1,2,\\ j=1,2,3}} (2q - 12i - 6l + 2j + 11), \max_{\substack{1 \le i \le k,\\ j=1,2,3}} (12k - 12i + 2j - 1)\}$$

$$= \max\{\max\{0, 12, 24 \ldots 12k\}, \max_{j=1,2,3}\{4j - 2, 4j + 10 \ldots 4j + 12k - 14\}, \max_{\substack{l=1,2,\\ j=1,2,3}}\{2q - 6l + 2j - 1, 2q - 6l + 2j - 13 \ldots 2q - 6l + 2j - 12k + 11\}, \max_{j=1,2,3}\{12k + 2j - 13, 12k + 2j - 25 \ldots 2j - 1\}\}$$

$$= 2q - 1$$

Hence, we find that :-

$$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$$

• It's obvious that $\forall\ u, v \in V(G) \ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $|\phi(w_{i1}^{1}) - \phi(u_i)| = \{2q - 24i + 19 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 29 \ldots 2q - 24k + 19\} \quad , 2q = 36k$

The range of $|\phi(w_{i1}^{1}) - \phi(w_i^{1})| = \{2q - 24i + 17 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q - 31 \ldots 2q - 24k + 17\} \quad , 2q = 36k$

The range of $|\phi(w_{i1}^{2}) - \phi(u_i)| = \{2q - 24i + 21 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 27 \ldots 2q - 24k + 21\} \quad , 2q = 36k$

The range of $|\phi(w_{i1}^{2}) - \phi(w_i^{2})| = \{2q - 24i + 15 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 9, 2q - 33 \ldots 2q - 24k + 15\} \quad , 2q = 36k$

The range of $|\phi(w_{i1}^{3}) - \phi(u_i)| = \{2q - 24i + 23 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 25 \ldots 2q - 24k + 23\} \quad , 2q = 36k$

The range of $|\phi(w_{i1}^{3}) - \phi(w_i^{3})| = \{2q - 24i + 13 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 11, 2q - 35 \ldots 2q - 24k + 13\} \quad , 2q = 36k$

The range of $|\phi(w_{i2}^{1}) - \phi(u_{i+1})| = \{2q - 24i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 23, 2q - 47 \ldots 2q - 24k + 1\} \quad , 2q = 36k$

The range of $|\phi(w_{i2}^{1}) - \phi(w_i^{1})| = \{2q - 24i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 13, 2q - 37 \ldots 2q - 24k + 11\} \quad , 2q = 36k$

The range of $|\phi(w_{i2}^{2}) - \phi(u_{i+1})| = \{2q - 24i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 21, 2q - 45 \ldots 2q - 24k + 3\} \quad , 2q = 36k$

The range of $| \phi(w_{i2}^2) - \phi(w_i^2) | = \{2q - 24i + 9 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 15, 2q - 39 \ldots 2q - 24k + 9 \} \quad , 2q = 36k$

The range of $| \phi(v_i^1) - \phi(u_i) | = \{12k - 24i + 13 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 11, 12k - 35 \ldots 13 - 12k\} \quad , 2q = 36k$

The range of $| \phi(v_i^2) - \phi(u_i) | = \{12k - 24i + 15 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 9, 12k - 33 \ldots 15 - 12k\} \quad , 2q = 36k$

The range of $| \phi(v_i^3) - \phi(u_i) | = \{12k - 24i + 17 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 7, 12k - 31 \ldots 17 - 12k\} \quad , 2q = 36k$

The range of $| \phi(v_i^1) - \phi(u_{i+1}) | = \{12k - 24i + 1 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 23, 12k - 47 \ldots 1 - 12k\} \quad , 2q = 36k$

The range of $| \phi(v_i^2) - \phi(u_{i+1}) | = \{12k - 24i + 3 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 21, 12k - 45 \ldots 3 - 12k\} \quad , 2q = 36k$

The range of $| \phi(v_i^3) - \phi(u_{i+1}) | = \{12k - 24i + 5 \quad , i = 1, 2 \ldots k \}$

$= \{12k - 19, 12k - 43 \ldots 5 - 12k\} \quad , 2q = 36k$

Hence, $\{ | \phi(u) - \phi(v) | : u v \in E(G)\} = \{ 1, 3, 5 \ldots 2q - 1\}$.

So the graph $(3, k)$ C_6 is odd graceful.

Example 5

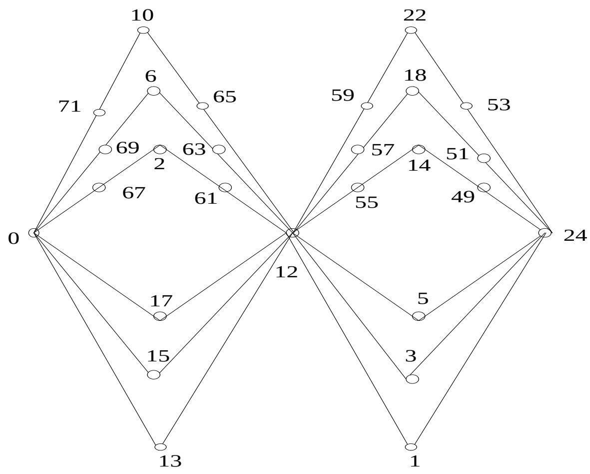

Figure 8: : The odd graceful labeling of the graph (3, 2) C_6

Theorem 4: The graph (m, k) C_6 is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, w_i^j, where i = 1, 2, 3…k, j = 1, 2, 3…m, v_i^j, where i = 1, 2, 3…k, j = 1, 2, 3…m and w_{il}^j, where i = 1, 2, 3…k , l = 1, 2, j = 1, 2, 3…m are the vertices of (m, k) C_6, such that v_i^j & w_i^j are put between u_i and $u_{i+1,}$ w_{i1}^j between u_i and w_p^j, such that i = 1, 2, 3…k+1, p = 1,2,3…k, l = 1, 2, j = 1, 2, 3…m, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1} and w_{i1}^j is beneath w_{i1}^{j+1} where j = 1, 2, 3…m , the graph (m, k) C_6 has number of edges "q" 6m k , as shown in the next figure.

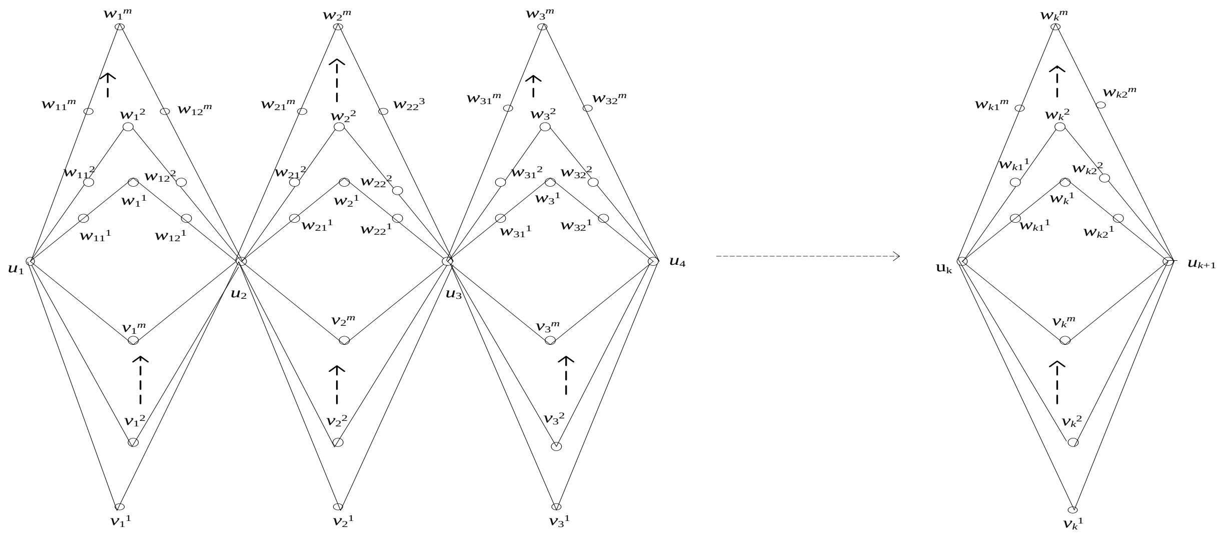

Figure 9: The graph (m, k) C_6

The number of edges "q"= 6m k

Define $\phi: V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 4m\,(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_i^j) = 4mi + 4j - 4m - 2$, $i = 1, 2, 3\ldots$ k , $j = 1, 2 \ldots m$

$\phi(w_{il}^j) = 2q - 4mi - 2ml + 2j + 4m - 1$, $i = 1, 2, 3 \ldots k$, $l = 1, 2,$ $j = 1, 2 \ldots m$

$\phi(v_i^j) = 4mk - 4mi + 2j - 1$, $i = 1, 2, 3\ldots k$, $j = 1, 2 \ldots m$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (m, k)\ C_6$

$$\exists \quad \max_{v \in V(G)} \phi(v) =$$

$$\max\{ \max_{1 \le i \le k+1} (4mi - 4m),\ \max_{\substack{1 \le i \le k,\\ 1 \le j \le m}} (4mi + 4j - 4m - 2),$$

$$\max_{\substack{1 \le i \le k\\ l=1,2,\\ 1 \le j \le m}} (2q - 4mi - 2ml + 2j + 4m - 1),$$

$$\max_{\substack{1 \le i \le k,\\ 1 \le j \le m}} (4mk - 4mi + 2j - 1)\}$$

$$= 2q - 1$$

Hence, we find that :−

$$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:
The range of $|\ \phi(w_{i1}^j) - \phi(u_i)\ | = \{2q - 8mi + 2j + 6m - 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

The range of $|\ \phi(w_{i1}^j) - \phi(w_i^j)\ | = \{2q - 8mi - 2j + 6m + 1$, $i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

The range of $|\phi(w_{i2}^{j}) - \phi(w_i^{j})| = \{2q - 8mi - 2j + 1 \quad , i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

The range of $|\phi(w_{i2}^{j}) - \phi(u_{i+1})| = \{2q - 8mi + 2j - 1 \quad , i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

The range of $|\phi(v_i^{j}) - \phi(u_i)| = \{4mk - 8mi + 2j + 4m - 1 \quad , i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

The range of $|\phi(v_i^{j}) - \phi(u_{i+1})| = \{4mk - 8mi + 2j - 1 \quad , i = 1, 2 \ldots k, j = 1, 2 \ldots m\}$

Hence, $\{ |\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph $(m, k)\ C_6$ is odd graceful.

Example 6

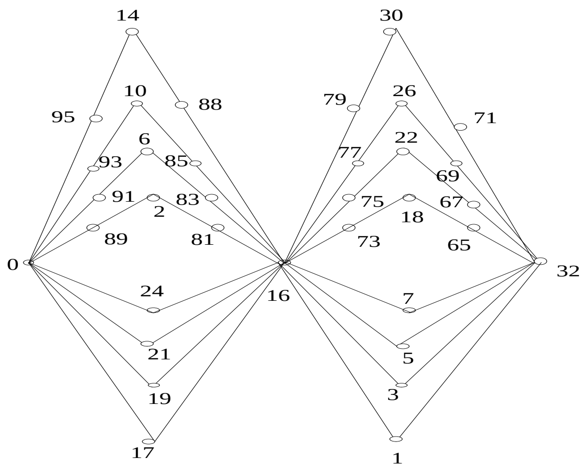

Figure 10: The odd graceful of the graph (4, 2) C_6

Theorem 1: The graph kC_8-snake " $(1, k)\ C_8$-snake" is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, $w_1w_2w_3\ldots w_k$, $v_1v_2v_3\ldots v_k$, $w_{11}w_{12}w_{21}\ w_{22}w_{31}w_{32}\ \ldots w_{k1}w_{k2}$ and $v_{11}v_{12}v_{21}\ v_{22}v_{31}v_{32}\ \ldots v_{k1}v_{k2}$ are the vertices of kC_8-snake, such that v_i & w_i are put between u_i and u_{i+1}, w_{il} is put between w_p and u_i such that $i = 1, 2, 3\ldots k+1$, $p = 1,2,3\ldots k$, $l = 1, 2$, v_{il} is put between v_p and u_i, where $i = 1, 2, 3\ldots k+1$, $p = 1,2,3\ldots k$, $l = 1, 2$, the graph kC_8-snake has number of edges "q" $8k$, as shown in the next figure.

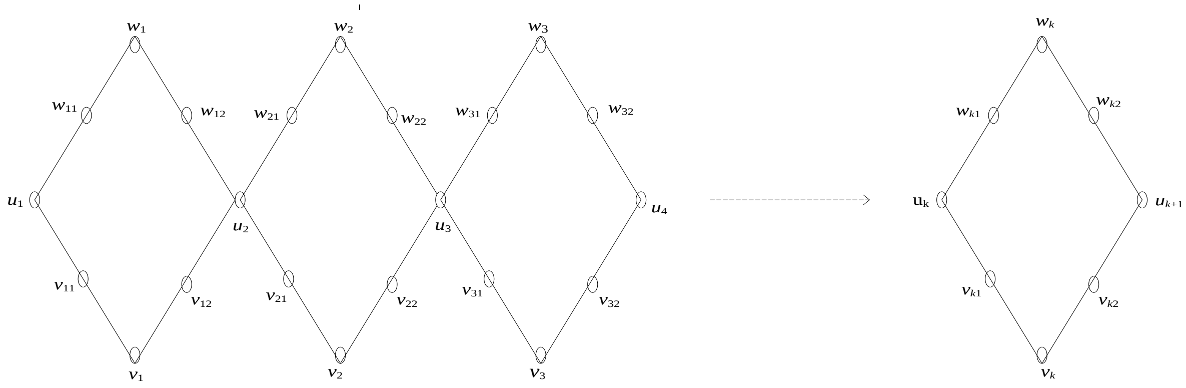

Figure 3: The graph kC_8-snake

The number of edges" q"= $8k$

Define $\phi : V(G) \to \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 8(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_i) = 8i - 2$, $i = 1, 2, 3\ldots k$

$\phi(v_i) = 8i - 6$, $i = 1, 2, 3\ldots k$

$\phi(w_{il}) = 2q - 8i - 4l + 11$, $i = 1, 2, 3\ldots k$, $l = 1, 2$

$\phi(v_{il}) = 2q - 8i - 4l + 9$, $i = 1, 2, 3\ldots k$, $l = 1, 2$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_8$-snake

$\exists$ $\max_{v \in V(G)} \phi(v) =$

$$\max\{ \max_{1\le i\le k+1}(8i-8),\ \max_{1\le i\le k}(8i-2),$$

$$\max_{\substack{1\le i\le k\\ l=1,2}}(2q-8i-4l+11),\ \max_{1\le i\le k}(8i-6),$$

$$\max_{\substack{1\le i\le k\\ l=1,2}}(2q-8i-4l+9)\}$$

$= \max\{\max\{0, 8, 16 \ldots 8k\}, \max\{6, 14 \ldots 8k - 2\}, \max_{l=1,2}\{2q-4l+3, 2q-4l-5 \ldots 2q-4l-8k+11\}$, max{2, 10 … 8k - 6}, $\max_{l=1,2}\{2q-4l+1, 2q-4l-7 \ldots 2q-4l-8k+9\}\}$

= $2q$ - 1

Hence, we find that :−

$\phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

• It's obvious that $\forall\, u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

∴ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set { 0,1,2…$2q$ - 1}

• Now, we want to show that the labels of the edges of G belong to the set {1, 3, 5 … $2q$ - 1 } and that's as following:

The range of $|\phi(w_{i1}) - \phi(u_i)| = \{2q-16i+15$, $i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q-17 \ldots 2q-16k+15\}$, $2q = 16k$

$=\{2q-1, 2q-17 \ldots 15\}$

The range of $|\phi(w_{i1}) - \phi(w_i)| = \{2q-16i+9$, $i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q-23 \ldots 2q-16k+9\}$, $2q = 16k$

$= \{2q - 7, 2q-23 \ldots 9\}$

The range of $|\phi(w_{i2}) - \phi(w_i)| = \{2q-16i+5$, $i = 1, 2 \ldots k\}$

$= \{2q - 11, 2q-27 \ldots 2q-16k+5\}$, $2q = 16k$

$= \{2q - 11, 2q-27 \ldots 5\}$

The range of $|\phi(w_{i2}) - \phi(u_{i+1})| = \{2q-16i+3$, $i = 1, 2 \ldots k\}$

$= \{2q - 13, 2q-29 \ldots 2q-16k+3\}$, $2q = 16k$

$= \{2q - 13, 2q-29 \ldots 3\}$

The range of $|\phi(v_{i1}) - \phi(u_i)| = \{2q-16i+13$, $i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q-19 \ldots 2q-16k+13\}$, $2q = 16k$

$=\{2q - 3, 2q - 19 \ldots 13\}$

The range of $|\phi(v_{i1}) - \phi(v_i)| = \{2q - 16i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 5, 2q - 21 \ldots 2q - 16k + 11\} \quad , 2q = 16k$

$= \{2q - 5, 2q - 21 \ldots 11\}$

The range of $|\phi(v_{i2}) - \phi(v_i)| = \{2q - 16i + 7 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 9, 2q - 25 \ldots 2q - 16k + 7\} \quad , 2q = 16k$

$= \{2q - 9, 2q - 25 \ldots 7\}$

The range of $|\phi(v_{i2}) - \phi(u_{i+1})| = \{2q - 16i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 15, 2q - 31 \ldots 2q - 16k + 1\} \quad , 2q = 16k$

$= \{2q - 15, 2q - 31 \ldots 1\}$

Hence, $\{ |\phi(u) - \phi(v)| : u\,v \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph kC_8-snake is odd graceful.

Example 3

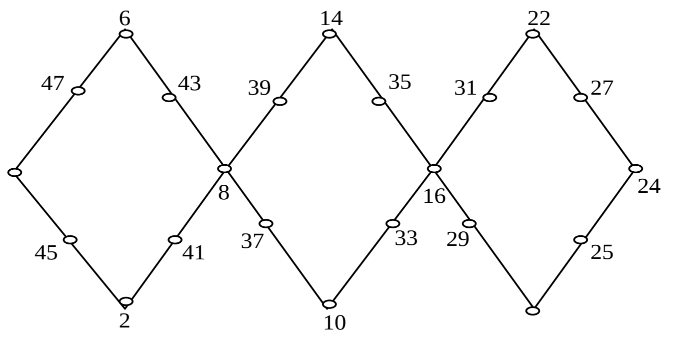

Figure 4: The odd graceful labeling of the graph $3C_8$-snake

Theorem 2: The graph (2, k) C_8-snake is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}$, w_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2$, v_i^j, where $i = 1, 2, 3\ldots k$, $j = 1, 2$, w_{il}^j, where $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2$ and v_{il}^j, where $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2$ are the vertices of (2, k) C_8-snake, such that v_i^j & w_i^j are put between u_i and $u_{i+1,}w_{il}^j$ is put between w_p^j and u_i such that $i = 1, 2, 3\ldots k+1$, $p = 1,2,3\ldots k$, $l = 1, 2$, $j = 1, 2$, v_{il}^j is put between v_p^j and u_i such that $i = 1, 2, 3\ldots k+1$, $p = 1,2,3\ldots k$, $l = 1, 2$, $j = 1, 2$, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1}, the graph (2, k) C_8-snake has number of edges "q" $16k$, as shown in the next figure.

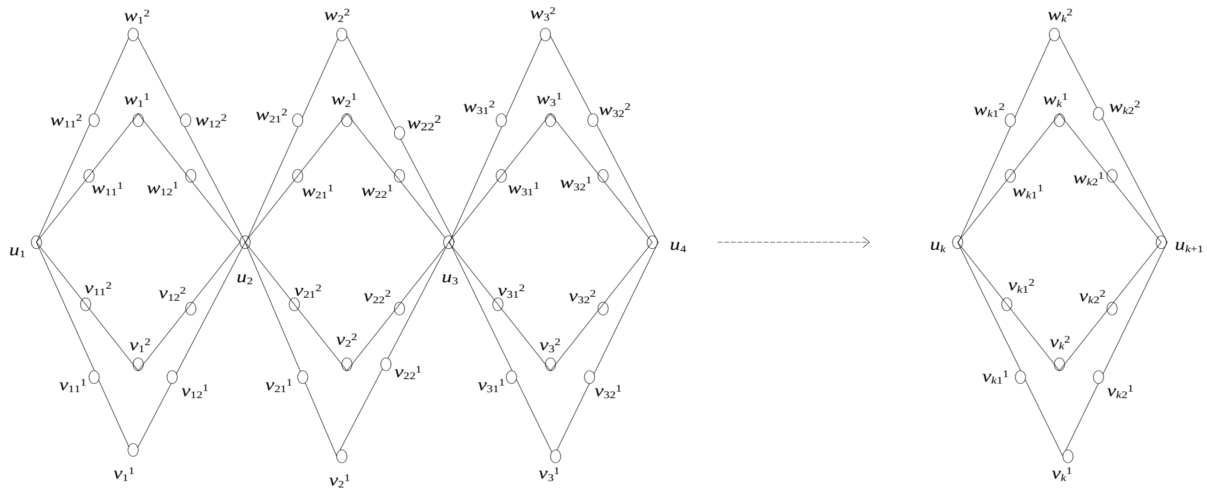

Figure 5: The graph (2, *k*) C_8-snake

The number of edges" q"= $16k$

Define $\phi: V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 16(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_i^j) = 16i + 4j - 10$, $i = 1, 2, 3\ldots k$, $j = 1, 2$

$\phi(v_i^j) = 16i + 4j - 18$, $i = 1, 2, 3\ldots k$, $j = 1, 2$

$\phi(w_{i1}^j) = 2q - 16i - 8l + 2j + 19$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2$

$\phi(\mathrm{v}_{i1}^j) = 2q - 16i - 8l + 2j + 15$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (2, k) C_8-snake

$\exists \max_{v \in V(G)} \phi(v) =$

$$\max\{ \max_{1 \le i \le k+1} (16i - 16), \max_{\substack{1 \le i \le k, \\ j=1,2}} (16i + 4j - 10),$$

$$\max_{\substack{1 \le i \le k \\ l=1,2 \\ j=1,2}} (2q - 16i - 8l + 2j + 19), \max_{\substack{1 \le i \le k, \\ j=1,2}} (16i + 4j - 18),$$

$$\max_{\substack{1 \le i \le k \\ l=1,2, \\ j=1,2}} (2q - 16i - 8l + 2j + 15)\}$$

$$= \max\{\max\{0, 16, 32 \ldots 16k\}, \max_{j=1,2}\{4j + 6, 4j + 22 \ldots 4j + 16k - 10\},$$

$$\max_{\substack{l=1,2, \\ j=1,2}}\{2q - 8l + 2j + 3, 2q - 8l + 2j - 13 \ldots 2q - 8l + 2j - 16k + 19\}$$

$$, \max_{j=1,2}\{4j - 2, 4j + 14 \ldots 4j + 16k - 18\},$$

$$\max_{\substack{l=1,2, \\ j=1,2}}\{2q - 8l + 2j - 1, 2q - 8l + 2j - 17 \ldots 2q - 8l + 2j - 16k + 15\}\}$$

$= 2q - 1$

Hence, we find that :−

$\phi(v) \in \{ 0, 1, 2 \ldots 2q - 1\}$

• It's obvious that $\forall\, u, v \in V(G) \exists\, \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2 \ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $| \phi(w_{i1}^{1}) - \phi(u_i) | = \{2q - 32i + 29 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q - 35 \ldots 2q - 32k + 29\} \quad , 2q = 32k$

$= \{2q - 3, 2q - 35 \ldots 29\}$

The range of $|\phi(w_{i1}^1) - \phi(w_i^1)| = \{2q - 32i + 19 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 13, 2q - 45 \ldots 2q - 32k + 19\} \quad , 2q = 32k$

$= \{2q - 13, 2q - 45 \ldots 19\}$

The range of $|\phi(w_{i2}^1) - \phi(w_i^1)| = \{2q - 32i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 21, 2q - 53 \ldots 2q - 32k + 11\} \quad , 2q = 32k$

$= \{2q - 21, 2q - 53 \ldots 11\}$

The range of $|\phi(w_{i2}^1) - \phi(u_{i+1})| = \{2q - 32i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 27, 2q - 59 \ldots 2q - 32k + 5\} \quad , 2q = 32k$

$= \{2q - 27, 2q - 59 \ldots 5\}$

The range of $|\phi(w_{i1}^2) - \phi(u_i)| = \{2q - 32i + 31 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 1, 2q - 33 \ldots 2q - 32k + 31\} \quad , 2q = 32k$

$= \{2q - 1, 2q - 33 \ldots 31\}$

The range of $|\phi(w_{i1}^2) - \phi(w_i^2)| = \{2q - 32i + 17 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 15, 2q - 47 \ldots 2q - 32k + 17\} \quad , 2q = 32k$

$= \{2q - 15, 2q - 47 \ldots 17\}$

The range of $|\phi(w_{i2}^2) - \phi(w_i^2)| = \{2q - 32i + 9 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 23, 2q - 55 \ldots 2q - 32k + 9\} \quad , 2q = 32k$

$= \{2q - 23, 2q - 55 \ldots 9\}$

The range of $|\phi(w_{i2}^2) - \phi(u_{i+1})| = \{2q - 32i + 7 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 25, 2q - 57 \dots 2q - 32k + 7\}$, $2q = 32k$

$= \{2q - 25, 2q - 57 \dots 7\}$

The range of $|\phi(v_{i1}^1) - \phi(u_i)| = \{2q - 32i + 25$, $i = 1, 2 \dots k\}$

$= \{2q - 7, 2q - 39 \dots 2q - 32k + 25\}$, $2q = 32k$

$= \{2q - 7, 2q - 39 \dots 25\}$

The range of $|\phi(v_{i1}^1) - \phi(v_i^1)| = \{2q - 32i + 23$, $i = 1, 2 \dots k\}$

$= \{2q - 9, 2q - 41 \dots 2q - 32k + 23\}$, $2q = 32k$

$= \{2q - 9, 2q - 41 \dots 23\}$

The range of $|\phi(v_{i2}^1) - \phi(v_i^1)| = \{2q - 32i + 15$, $i = 1, 2 \dots k\}$

$= \{2q - 17, 2q - 49 \dots 2q - 32k + 15\}$, $2q = 32k$

$= \{2q - 17, 2q - 49 \dots 15\}$

The range of $|\phi(v_{i2}^1) - \phi(u_{i+1})| = \{2q - 32i + 1$, $i = 1, 2 \dots k\}$

$= \{2q - 31, 2q - 63 \dots 2q - 32k + 1\}$, $2q = 32k$

$= \{2q - 31, 2q - 63 \dots 1\}$

The range of $|\phi(v_{i1}^2) - \phi(u_i)| = \{2q - 32i + 27$, $i = 1, 2 \dots k\}$

$= \{2q - 5, 2q - 37 \dots 2q - 32k + 27\}$, $2q = 32k$

$= \{2q - 5, 2q - 37 \dots 27\}$

The range of $|\phi(v_{i1}^2) - \phi(v_i^2)| = \{2q - 32i + 21$, $i = 1, 2 \dots k\}$

$= \{2q - 11, 2q - 43 \dots 2q - 32k + 21\}$, $2q = 32k$

$= \{2q - 11, 2q - 43 \dots 21\}$

The range of $| \phi(v_{i2}^2) - \phi(v_i^2) | = \{2q - 32i + 13 \quad , i = 1, 2 \dots k \}$

$= \{2q - 19, 2q - 51 \dots 2q - 32k + 13 \} \quad , 2q = 32k$

$= \{2q - 19, 2q - 51 \dots 13 \}$

The range of $| \phi(v_{i2}^2) - \phi(u_{i+1}) | = \{2q - 32i + 3 \quad , i = 1, 2 \dots k \}$

$= \{2q - 29, 2q - 61 \dots 2q - 32k + 3 \} \quad , 2q = 32k$

$= \{2q - 29, 2q - 61 \dots 3 \}$

Hence, $\{ | \phi(u) - \phi(v) | : uv \in E(G)\} = \{ 1, 3, 5 \dots 2q - 1\}$.

So the graph (2, k) C_8-snake is odd graceful.

Example 4

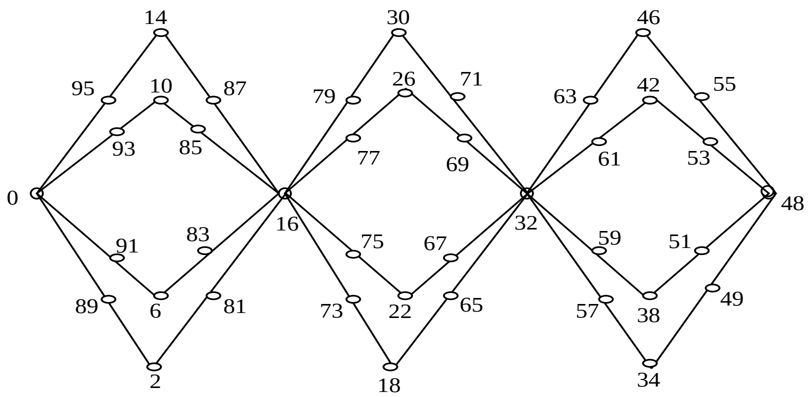

Figure 6: The odd graceful labeling of the graph (2, 3) C_8-snake

Theorem 3: The graph (3, k) C_8-snake is odd graceful.

Proof

Let $u_1u_2u_3 \dots u_{k+1}$, w_i^j $i = 1, 2, 3 \dots k$, $j = 1, 2, 3$, v_i^j $i = 1, 2, 3 \dots k$, $j = 1, 2, 3$, w_{il}^j, $i = 1, 2, 3 \dots k$, $l = 1, 2$, $j = 1, 2, 3$ and v_{il}^j, $i = 1, 2, 3 \dots k$, $l = 1, 2$, $j = 1, 2, 3$ are the vertices of (3, k) C_8-snake, such that v_i^j & w_i^j are put between u_i and u_{i+1},w_{il}^j is put between w_p^j and u_i such that $i = 1, 2, 3 \dots k+1$, $p = 1,2,3 \dots k$, $l = 1, 2$, $j = 1, 2, 3$, v_{il}^j is put between v_p^j and u_i such that $i = 1, 2, 3 \dots k+1$, $p = 1,2,3 \dots k$, $l = 1, 2$, $j = 1, 2, 3$, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1}, where $j = 1, 2, 3$, the graph (3, k) C_8-snake has number of edges "q" $24k$, as shown in the next figure.

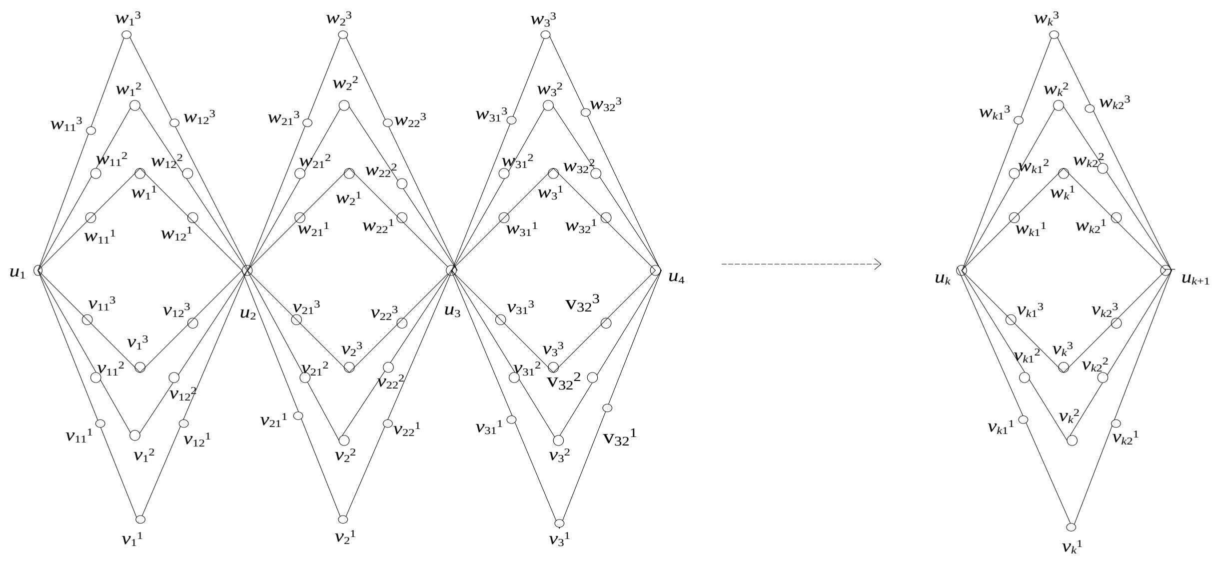

Figure 7: The graph (3, k) C_8-snake

The number of edges" q"= $24k$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 24(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_i^j) = 24i + 4j - 14$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$

$\phi(v_i^j) = 24i + 4k - 26$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3$

$\phi(w_{il}^j) = 2q - 24i - 12l + 2j + 29$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2, 3$

$\phi(v_{il}^j) = 2q - 24i - 12l + 2j + 23$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1,2, 3$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (3, k) C_8-snake

$\exists$ $\max_{v \in V(G)} \phi(v)$ =

$$\max\{ \max_{1 \le i \le k+1} (24i - 24),\ \max_{\substack{1 \le i \le k,\\ j=1,2,3}} (24i + 4j - 14),$$

$$\max_{\substack{1\le i\le k\\ l=1,2\\ j=1,2,3}}(2q-24i-12l+2j+29),\ \max_{\substack{1\le i\le k,\\ j=1,2,3}}(24i+4j-26),$$

$$\max_{\substack{1\le i\le k\\ l=1,2,\\ j=1,2,3}}(2q-24i-12l+2j+23)\}$$

$$= \max\{\max\{0, 24, 48 \ldots 24k\},\ \max_{j=1,2,3}\{4j+10, 4j+34 \ldots\ 4j+24k-14\},$$

$$\max_{\substack{l=1,2,\\ j=1,2,3}}\{2q-12l+2j+5, 2q-12l+2j-19 \ldots 2q-12l+2j-24k+29\}$$

$$,\ \max_{j=1,2,3}\{4j-2, 4j+22 \ldots 4j+24k-26\},$$

$$\max_{\substack{l=1,2,\\ j=1,2,3}}\{2q-12l+2j-1, 2q-12l+2j-25 \ldots 2q-12l+2j-24k+23\}\}$$

$= 2q - 1$

Hence, we find that :−

$\phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

• It's obvious that $\forall\, u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q-1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q-1\}$ and that's as following:

The range of $|\phi(w_{i1}^{1}) - \phi(u_i)| = \{2q-48i+43 \quad , i = 1, 2 \ldots k\}$

$= \{2q-5, 2q-53 \ldots 2q-48k+43\} \quad , 2q = 48k$

$= \{2q-5, 2q-53 \ldots 43\}$

The range of $|\phi(w_{i1}^{1}) - \phi(w_i^{1})| = \{2q-48i+29 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 19, 2q – 67 \ldots 2q – 48k + 29\}$, $2q = 48k$

$= \{2q - 19, 2q – 67 \ldots 29\}$

The range of $| \phi(w_{i2}^1) - \phi(w_i^1) | = \{2q – 48i + 17$, $i = 1, 2 \ldots k\}$

$= \{2q - 31, 2q – 79 \ldots 2q – 48k + 17\}$, $2q = 48k$

$= \{2q - 31, 2q – 79 \ldots 17\}$

The range of $| \phi(w_{i2}^1) - \phi(u_{i+1}) | = \{2q – 48i + 7$, $i = 1, 2 \ldots k\}$

$= \{2q - 41, 2q – 89 \ldots 2q – 48k + 7\}$, $2q = 48k$

$= \{2q - 41, 2q – 89 \ldots 7\}$

The range of $| \phi(w_{i1}^2) - \phi(u_i) | = \{2q – 48i + 45$, $i = 1, 2 \ldots k\}$

$= \{2q - 3, 2q – 51 \ldots 2q – 48k + 45\}$, $2q = 48k$

$=\{2q – 3, 2q – 51 \ldots 45\}$

The range of $| \phi(w_{i1}^2) - \phi(w_i^2) | = \{2q – 48i + 27$, $i = 1, 2 \ldots k\}$

$= \{2q - 21, 2q – 69 \ldots 2q – 48k + 27\}$, $2q = 48k$

$= \{2q - 21, 2q – 69 \ldots 27\}$

The range of $| \phi(w_{i2}^2) - \phi(w_i^2) | = \{2q – 48i + 15$, $i = 1, 2 \ldots k\}$

$= \{2q - 33, 2q – 81 \ldots 2q – 48k + 15\}$, $2q = 48k$

$= \{2q - 33, 2q – 81 \ldots 15\}$

The range of $| \phi(w_{i2}^2) - \phi(u_{i+1}) | = \{2q – 48i + 9$, $i = 1, 2 \ldots k\}$

$= \{2q - 39, 2q – 87 \ldots 2q – 48k + 9\}$, $2q = 48k$

$= \{2q - 39, 2q – 87 \ldots 9\}$

The range of $| \phi(w_{i1}^3) - \phi(u_i) | = \{2q – 48i + 47$, $i = 1, 2 \ldots k \}$

$= \{2q - 1, 2q – 49 \ldots 2q – 48k + 47 \}$, $2q = 48k$

$=\{2q – 1, 2q – 49 \ldots 47\}$

The range of $| \phi(w_{i1}^3) - \phi(w_i^3) | = \{2q – 48i + 25$, $i = 1, 2 \ldots k \}$

$= \{2q - 23, 2q – 71 \ldots 2q – 48k + 25\}$, $2q = 48k$

$= \{2q - 23, 2q – 71 \ldots 25\}$

The range of $| \phi(w_{i2}^3) - \phi(w_i^3) | = \{2q – 48i + 13$, $i = 1, 2 \ldots k \}$

$= \{2q - 35, 2q – 83 \ldots 2q – 48k + 13\}$, $2q = 48k$

$= \{2q - 35, 2q – 83 \ldots 13\}$

The range of $| \phi(w_{i2}^3) - \phi(u_{i+1}) | = \{2q – 48i + 11$, $i = 1, 2 \ldots k \}$

$= \{2q - 37, 2q – 85 \ldots 2q – 48k + 11\}$, $2q = 48k$

$= \{2q - 37, 2q – 85 \ldots 11\}$

The range of $| \phi(v_{i1}^1) - \phi(u_i) | = \{2q – 48i + 37$, $i = 1, 2 \ldots k \}$

$= \{2q - 11, 2q – 59 \ldots 2q – 48k + 37 \}$, $2q = 48k$

$=\{2q – 11, 2q – 59 \ldots 37\}$

The range of $| \phi(v_{i1}^1) - \phi(v_i^1) | = \{2q – 48i + 35$, $i = 1, 2 \ldots k \}$

$= \{2q - 13, 2q – 61 \ldots 2q – 48k + 35 \}$, $2q = 48k$

$= \{2q - 13, 2q – 61 \ldots 35 \}$

The range of $| \phi(v_{i2}^1) - \phi(v_i^1) | = \{2q – 48i + 23$, $i = 1, 2 \ldots k \}$

$= \{2q - 25, 2q - 73 \ldots 2q - 48k + 23\}$, $2q = 48k$

$= \{2q - 25, 2q - 73 \ldots 23\}$

The range of $|\phi(v_{i2}^1) - \phi(u_{i+1})| = \{2q - 48i + 1$, $i = 1, 2 \ldots k\}$

$= \{2q - 47, 2q - 95 \ldots 2q - 48k + 1\}$, $2q = 48k$

$= \{2q - 47, 2q - 95 \ldots 1\}$

The range of $|\phi(v_{i1}^2) - \phi(u_i)| = \{2q - 48i + 39$, $i = 1, 2 \ldots k\}$

$= \{2q - 9, 2q - 57 \ldots 2q - 48k + 39\}$, $2q = 48k$

$= \{2q - 9, 2q - 57 \ldots 39\}$

The range of $|\phi(v_{i1}^2) - \phi(v_i^2)| = \{2q - 48i + 33$, $i = 1, 2 \ldots k\}$

$= \{2q - 15\ 2q - 63 \ldots 2q - 48k + 33\}$, $2q = 48k$

$= \{2q - 15, 2q - 63 \ldots 33\}$

The range of $|\phi(v_{i2}^2) - \phi(v_i^2)| = \{2q - 48i + 21$, $i = 1, 2 \ldots k\}$

$= \{2q - 27, 2q - 75 \ldots 2q - 48k + 21\}$, $2q = 48k$

$= \{2q - 27, 2q - 75 \ldots 21\}$

The range of $|\phi(v_{i2}^2) - \phi(u_{i+1})| = \{2q - 48i + 3$, $i = 1, 2 \ldots k\}$

$= \{2q - 45, 2q - 93 \ldots 2q - 48k + 3\}$, $2q = 48k$

$= \{2q - 45, 2q - 93 \ldots 3\}$

The range of $|\phi(v_{i1}^3) - \phi(u_i)| = \{2q - 48i + 41$, $i = 1, 2 \ldots k\}$

$= \{2q - 7, 2q - 55 \ldots 2q - 48k + 41\}$, $2q = 48k$

$= \{2q - 7, 2q - 55 \ldots 41\}$

The range of $|\phi(v_{i1}^3) - \phi(v_i^3)| = \{2q - 48i + 31 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 17, 2q - 65 \ldots 2q - 48k + 31\} \quad , 2q = 48k$

$= \{2q - 17, 2q - 65 \ldots 31\}$

The range of $|\phi(v_{i2}^3) - \phi(v_i^3)| = \{2q - 48i + 19 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 29, 2q - 77 \ldots 2q - 48k + 19\} \quad , 2q = 48k$

$= \{2q - 29, 2q - 77 \ldots 19\}$

The range of $|\phi(v_{i2}^3) - \phi(u_{i+1})| = \{2q - 48i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{2q - 43, 2q - 91 \ldots 2q - 48k + 5\} \quad , 2q = 48k$

$= \{2q - 43, 2q - 91 \ldots 5\}$

Hence, $\{ |\phi(u) - \phi(v)| : \text{u v} \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the graph $(3, k)$ C_8-snake is odd graceful.

Example 5

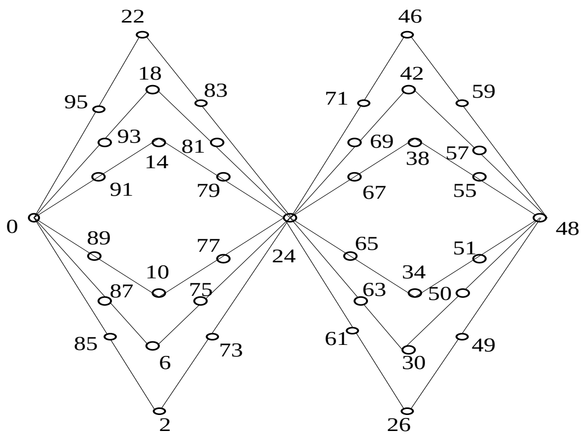

Figure 8: The odd graceful labeling of the graph (3, 2) C_8-snake

Theorem 4: The graph (m, k) C_8-snake is odd graceful.

Proof

Let $u_1u_2u_3…u_{k+1}$, w_i^j $i = 1, 2, 3…k$, $j = 1, 2, 3…m$, v_i^j $i = 1, 2, 3…k$, $j = 1, 2, 3…m$, w_{il}^j, $i = 1, 2, 3…k$, $l = 1, 2$, $j = 1, 2, 3…m$ and v_{il}^j, $i = 1, 2, 3…k$, $l = 1, 2$, $j = 1, 2, 3…m$ are the vertices of (m, k) C_8-snake, such that v_i^j & w_i^j are put between u_i and $u_{i+1,}w_{il}^j$ is put between w_p^j and u_i such that $i = 1, 2, 3…k+1$, $p = 1,2,3…k$, $l = 1, 2$, $j = 1, 2, 3…m$, v_{il}^j is put between v_p^j and u_i such that $i = 1, 2, 3…k+1$, $p = 1,2,3…k$, $l = 1, 2$, $j = 1, 2, 3…m$, v_i^j is beneath v_i^{j+1}, w_i^j is beneath w_i^{j+1}, where $j = 1, 2, 3…m$, the graph (m, k) C_8-snake has number of edges "q" $8m\,k$, as shown in the next figure.

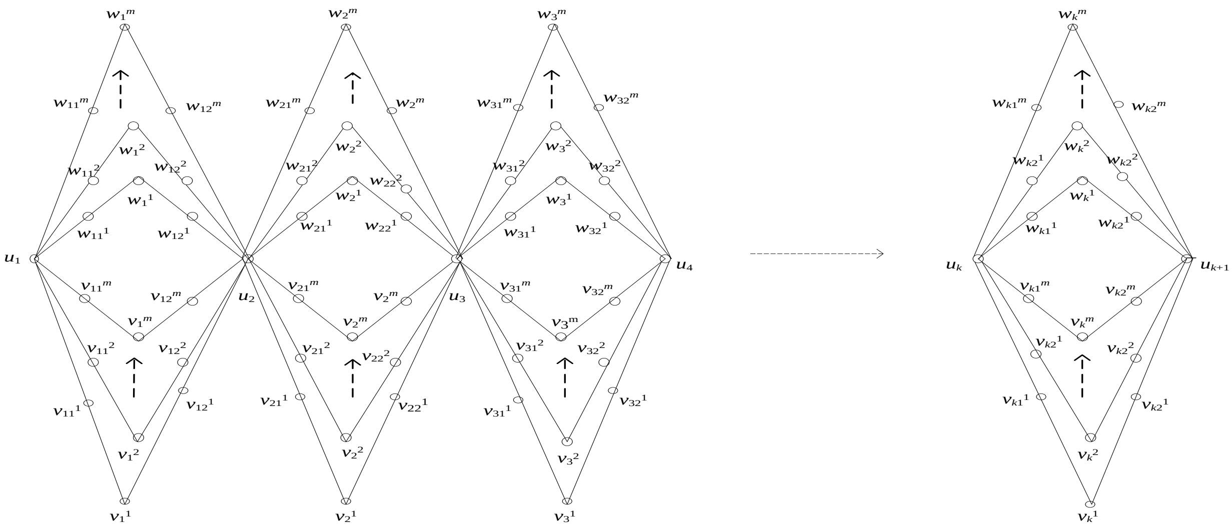

Figure 9: The graph (m, k) C_8-snake

The number of edges" q"= $8m\,k$

Define $\phi : V(G) \rightarrow \{0, 1, 2… 2q-1\}$ as following:

$\phi(u_i) = 8m(i-1)$, $i = 1, 2, 3… k+1$

$\phi(w_i^j) = 8mi + 4j - 4m - 2$, $i = 1, 2, 3… k$, $j = 1, 2, 3…m$

$\phi(v_i^j) = 8mi + 4j - 8m - 2$, $i = 1, 2, 3… k$, $j = 1, 2, 3…m$

$\phi(w_{i1}^j) = 2q - 8mi - 4ml + 2j + 10m - 1$, $i = 1, 2, 3… k$, $l = 1, 2$, $j = 1, 2, 3…m$

$\phi(v_{i1}^{j}) = 2q - 8mi - 4ml + 2j + 8m - 1$, $i = 1, 2, 3\ldots k$, $l = 1, 2$, $j = 1, 2, 3\ldots m$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (m, k)$ C_8-snake

$$\exists \max_{v\in V(G)} \phi(v) =$$

$$\max\{ \max_{1\le i\le k+1} (8mi - 8m), \max_{\substack{1\le i\le k,\\ 1\le j\le m}}(8mi + 4j - 4m - 2),$$

$$\max_{\substack{1\le i\le k\\ l=1,2\\ 1\le j\le m}}(2q - 8mi - 4ml + 2j + 10m - 1),$$

$$\max_{\substack{1\le i\le k,\\ 1\le j\le m}}(8mi + 4j - 8m - 2),$$

$$\max_{\substack{1\le i\le k\\ l=1,2,\\ 1\le j\le m}}(2q - 8mi - 4ml + 2j + 8m - 1)\}$$

$$= \max\{\max\{0, 8m, 16m \ldots 8mk\},$$

$$\max_{1\le j\le m}\{4j + 4m - 2, 4j + 12m - 2 \ldots 4j + 8mk - 4m - 2\},$$

$$\max_{\substack{l=1,2,\\ 1\le j\le m}}\{2q - 4ml + 2j + 2m - 1, 2q - 4ml + 2j - 6m - 1 \ldots 2q - 4ml + 2j - 8mk + 10m - 1\}$$

$$, \max_{1\le j\le m}\{4j - 2, 4j + 8m - 2 \ldots 4j + 8mk - 8m - 2\},$$

$$\max_{\substack{l=1,2,\\ j=1,2,3}}\{2q - 4ml + 2j - 1, 2q - 4ml + 2j - 8m - 1 \ldots 2q - 4ml + 2j - 8mk + 8m - 1\}\}$$

$$= 2q - 1$$

Hence, we find that :–

$$\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set { 0,1,2…2q - 1}

• Now, we want to show that the labels of the edges of G belong to the set {1, 3, 5 … 2q - 1 } and that's as following:

The range of $|\phi(w_{i1}^{j}) - \phi(u_i)| = \{2q - 16mi + 2j + 14m - 1$, $i = 1, 2 \dots k, j = 1, 2 \dots m\}$

$= \{2q + 2j - 2m - 1, 2q + 2j - 18m - 1 \dots 2q + 2j - 16mk + 14m - 1, j = 1, 2 \dots m\}$, $2q = 16mk$

The range of $|\phi(w_{i1}^{j}) - \phi(w_{i}^{j})| = \{2q - 16mi - 2j + 10m + 1$, $i = 1, 2 \dots k, j = 1, 2 \dots m\}$

$= \{2q - 2j - 6m + 1, 2q - 2j - 22m + 1 \dots 2q - 2j - 16mk + 10m + 1, j = 1, 2 \dots m\}$, $2q = 16mk$

The range of $|\phi(w_{i2}^{j}) - \phi(w_{i}^{j})| = \{2q - 16mi - 2j + 6m + 1$, $i = 1, 2 \dots k, j = 1, 2 \dots m\}$

$= \{2q - 2j - 10m + 1, 2q - 2j - 26m + 1 \dots 2q - 2j - 16mk + 6m + 1, j = 1, 2 \dots m\}$, $2q = 16mk$

The range of $|\phi(w_{i2}^{j}) - \phi(u_{i+1})| = \{2q - 16mi + 2j + 10m - 1$, $i = 1, 2 \dots k, j = 1, 2 \dots m\}$

$= \{2q + 2j - 6m - 1, 2q + 2j - 22m - 1 \dots 2q + 2j - 16mk + 10m - 1, j = 1, 2 \dots m\}$, $2q = 16mk$

The range of $|\phi(v_{i1}^{j}) - \phi(u_i)| = \{2q - 16mi + 2j + 12m - 1$, $i = 1, 2 \dots k, j = 1, 2 \dots m\}$

$= \{2q + 2j - 4m - 1, 2q + 2j - 20m - 1 \dots 2q + 2j - 16mk + 12m - 1, j = 1, 2 \dots m\}$, $2q = 16mk$

The range of | $\phi(v_{i1}^{j}) - \phi(v_i^j)$ | = {$2q - 16mi - 2j + 12m + 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots m$ }

= {$2q - 2j - 4m + 1$, $2q - 2j - 20m + 1 \ldots 2q - 2j - 16mk + 12m + 1$, $j = 1, 2 \ldots m$ } , $2q = 16mk$

The range of | $\phi(v_{i2}^{j}) - \phi(v_i^j)$ | = {$2q - 16mi - 2j + 8m + 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots m$ }

= {$2q - 2j - 8m + 1$, $2q - 2j - 24m + 1 \ldots 2q - 2j - 16mk + 8m + 1$, $j = 1, 2 \ldots m$ } , $2q = 16mk$

The range of | $\phi(v_{i2}^{j}) - \phi(u_{i+1})$ | = {$2q - 16mi + 2j + 8m - 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots m$ }

= {$2q + 2j - 8m - 1$, $2q + 2j - 24m - 1 \ldots 2q + 2j - 16mk + 8m - 1$, $j = 1, 2 \ldots m$ } , $2q = 16mk$

Hence, { | $\phi(u) - \phi(v)$ | : u v $\in$ E(G)} = { 1, 3, 5 … $2q - 1$}.

So the graph (m, k) C_8-snake is odd graceful.

Example 6

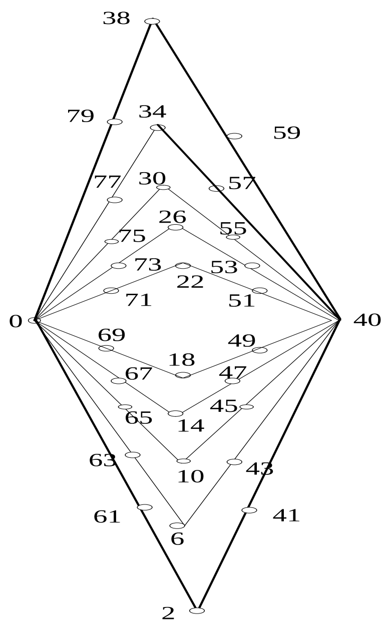

Figure 10: The odd graceful labeling of the graph (5, 1) C_8-snake

Theorem 5: The graph kC_{12}-snake is odd graceful.

Proof

Let $u_1u_2u_3\ldots u_{k+1}, w_1w_2w_3\ldots w_k$, $v_1v_2v_3\ldots v_k$, $a_1a_2a_3\ldots a_k$, $x_1x_2x_3\ldots x_k$, $b_1b_2b_3\ldots b_k$, $h_1h_2h_3\ldots h_k$, $c_1c_2c_3\ldots c_k$, $z_1z_2z_3\ldots z_k$, $d_1d_2d_3\ldots d_k$ and $g_1g_2g_3\ldots g_k$ are the vertices of kC_{12}-snake, such that v_i & w_i are put between u_i and u_{i+1}, a_i is put between w_i and u_i, x_i is put between a_i and u_i, b_i is put between w_i and u_{i+1}, h_i is put between b_i and u_{i+1}, c_i is put between v_i and u_i, z_i is put between c_i and u_i, d_i is put between v_i and u_{i+1}, g_i is put between d_i and u_{i+1} where i = 1, 2, 3…k, the graph (m, k) C_8-snake has number of edges "q" $12k$, as shown in the next figure.

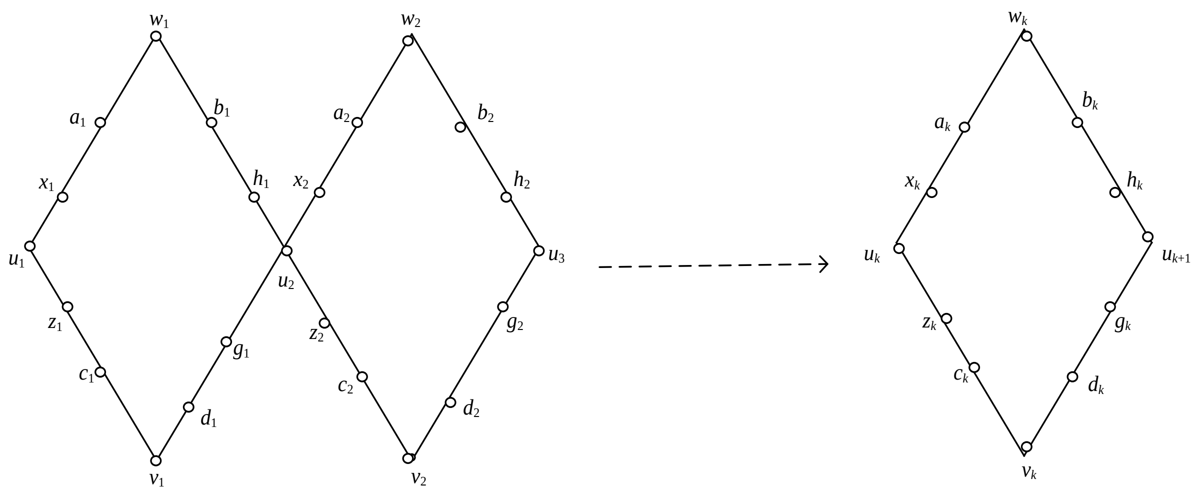

Figure 11: The graph kC_{12}-snake

The number of edges" q"= $12k$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u_i) = 6i - 6$, i = 1, 2, 3… k+1

$\phi(x_i) = 2q - 6i + 5$, i = 1, 2, 3… k

$\phi(h_i) = 2q - 6i + 1$, i = 1, 2, 3… k

$\phi(a_i) = 6i - 4$, $i = 1, 2, 3\ldots k$

$\phi(b_i) = 6i - 2$, $i = 1, 2, 3\ldots k$

$\phi(w_i) = 2q - 6i + 3$, $i = 1, 2, 3\ldots k$

$\phi(z_i) = q - 6i + 5$, $i = 1, 2, 3\ldots k$

$\phi(g_i) = q - 6i + 1$, $i = 1, 2, 3\ldots k$

$\phi(c_i) = 2q + 6i - 12k - 4$, $i = 1, 2, 3\ldots k$

$\phi(d_i) = 2q + 6i - 12k - 2$, $i = 1, 2, 3\ldots k$

$\phi(v_i) = q - 6i + 3$, $i = 1, 2, 3\ldots k$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_{12}$-snake

$$\exists \quad \max_{v \in V(G)} \phi(v) =$$

$$\max\{ \max_{1 \le i \le k+1} (6i - 6),\ \max_{1 \le i \le k}(2q - 6i + 5),\ \max_{1 \le i \le k}(6i - 4),\ \max_{1 \le i \le k}(2q - 6i + 1),\ \max_{1 \le i \le k}(6i - 2),\ \max_{1 \le i \le k}(2q - 6i + 3),\ \max_{1 \le i \le k}(q - 6i + 5),\ \max_{1 \le i \le k}(q - 6i + 1),\ \max_{1 \le i \le k}(2q + 6i - 12k - 4),\ \max_{1 \le i \le k}(2q + 6i - 12k - 2),\ \max_{1 \le i \le k}(q - 6i + 3) \}$$

$= \max\{\max\{0, 6, 12 \ldots 6k\}, \max\{2q - 1, 2q - 7 \ldots 2q - 6k + 5\}, \max\{2, 8, 14 \ldots 6k - 4\}, \max\{2q - 5, 2q - 11 \ldots 2q - 6k + 1\}, \max\{4, 10 \ldots 6k - 2\}, \max\{2q - 3, 2q - 9 \ldots 2q - 6k + 3\}, \max\{q - 1, q - 7 \ldots q - 6k + 5\}, \max\{q - 5, q - 11 \ldots q - 6k + 1\}, \max\{2q - 12k + 2, 2q - 12k + 8 \ldots 2q - 6k - 4\}, \max\{2q - 12k + 4, 2q - 12k + 10 \ldots 2q - 6k - 4\}, \max\{q - 3, q - 9 \ldots q - 6k + 3\}\}$

$= 2q - 1$

Hence, we find that :−

$\phi(v) \in \{ 0, 1, 2 \ldots 2q - 1\}$

• It's obvious that $\forall\, u, v \in V(G) \;\exists\; \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0,1,2\ldots 2q - 1\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 3, 5 \ldots 2q - 1\}$ and that's as following:

The range of $| \phi(x_i) - \phi(u_i) | = \{2q - 12i + 11 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 1, 2q - 13 \ldots 2q - 12k + 11 \} \quad , 2q = 24k$

$= \{2q - 1, 2q - 13 \ldots 11\}$

The range of $| \phi(x_i) - \phi(a_i) | = \{2q - 12i + 9 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 3, 2q - 15 \ldots 2q - 12k + 9 \} \quad , 2q = 24k$

The range of $| \phi(w_i) - \phi(a_i) | = \{2q - 12i + 7 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 5, 2q - 17 \ldots 2q - 12k + 7 \} \quad , 2q = 24k$

The range of $| \phi(w_i) - \phi(b_i) | = \{2q - 12i + 5 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 7, 2q - 19 \ldots 2q - 12k + 5 \} \quad , 2q = 24k$

The range of $| \phi(h_i) - \phi(b_i) | = \{2q - 12i + 3 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 9, 2q - 21 \ldots 2q - 12k + 3 \} \quad , 2q = 24k$

The range of $| \phi(h_i) - \phi(u_{i+1}) | = \{2q - 12i + 1 \quad , i = 1, 2 \ldots k \}$

$= \{2q - 11, 2q - 23 \ldots 2q - 12k + 1 \} \quad , 2q = 24k$

The range of $| \phi(z_i) - \phi(u_i) | = \{q - 12i + 11 \quad , i = 1, 2 \ldots k \}$

$= \{ q - 1, q - 13 \ldots q - 12k + 11 \} \quad , q = 12k$

$= \{ q - 1, q - 13 \ldots 11 \}$

The range of $| \phi(c_i) - \phi(z_i) | = \{q + 12i - 12k - 9 \quad , i = 1, 2 \ldots k \}$

$= \{ q - 12k + 3, q - 12k + 15 \ldots q - 9 \} \qquad , q = 12k$

$= \{3, 15 \ldots q - 9 \}$

The range of $| \phi(c_i) - \phi(v_i) | = \{q + 12i - 12k - 7 \qquad , i = 1, 2 \ldots k \}$

$= \{ q - 12k + 5, q - 12k + 17 \ldots q - 7 \} \qquad , q = 12k$

$= \{5, 17 \ldots q - 7 \}$

The range of $| \phi(d_i) - \phi(v_i) | = \{q + 12i - 12k - 5 \qquad , i = 1, 2 \ldots k \}$

$= \{ q - 12k + 7, q - 12k + 19 \ldots q - 5 \} \qquad , q = 12k$

$= \{7, 19 \ldots q - 5 \}$

The range of $| \phi(d_i) - \phi(g_i) | = \{q + 12i - 12k - 3 \qquad , i = 1, 2 \ldots k \}$

$= \{ q - 12k + 9, q - 12k + 21 \ldots q - 3 \} \qquad , q = 12k$

$= \{9, 21 \ldots q - 3 \}$

The range of $| \phi(g_i) - \phi(u_{i+1}) | = \{q - 12i + 7 \qquad , i = 1, 2 \ldots k \}$

$= \{ q - 5, q - 17 \ldots q - 12k + 7 \} \qquad , q = 12k$

$= \{ q - 5, q - 17 \ldots 7 \}$

Hence, $\{ | \phi(u) - \phi(v) | : uv \in E(G)\} = \{ 1, 3, 5 \ldots 2q - 1\}$.

So the graph kC_{12}-snake is odd graceful.

Example 7

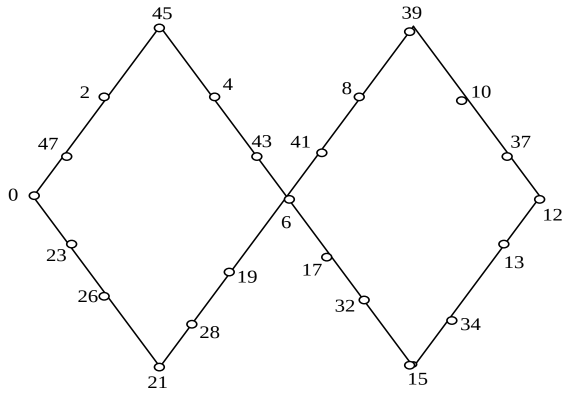

Figure 12: The odd graceful labeling of the graph $2C_{12}$-snake

Conclusion:

In this paper, we defined the graph (m, k) C_4-snake and proved that the graphs kC_4- snake is odd graceful, we proved the graph (2, k) C_4-snake is odd graceful, we introduced the odd graceful labeling of the graph (3, k) C_4-snake, we introduced the odd graceful labeling of the graph (m, k) C_4-snake**.** We proved that the graphs kC_6- snake is odd graceful, we proved the graph (2, k) C_6-snake is odd graceful, we introduced the odd graceful labeling of the graph (3, k) C_6-snake, we introduced the odd graceful labeling of the graph (m, k) C_6-snake**.** we proved that the graph (2, k) C_8-snake is odd graceful, we introduced the odd graceful labeling of the graph (3, k) C_8-snake, we introduced the odd graceful labeling of the graph (m, k) C_8-snake, and after that we proved that the graph kC_{12}-snake is odd graceful.

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